TECHNICAL UNIVERSITY OF LIBEREC
Faculty of Mechanical Engineering

prof. Ing. Miloslav Okrouhlík, CSc.

MECHANICS OF DEFORMABLE BODIES

2018

Reviewer: doc. Ing. Petr Hora, CSc.

ISBN 978-80-7494- 453-6

Mechanics of deformable bodies

00_SC. Scope

01_IN. Introductory part

1.1. Introduction
1.1.1. A few words about modeling
1.1.2. Continuum mechanics
1.1.3. Strength of material
1.1.4. Linear vs. non-linear
1.1.5. Sources of non-linearity
1.1.6. System of units
1.2. History of mechanics of rigid and deformable bodies
1.3. Mathematical and computational tools – backgrounds
1.3.1. Scalars, vectors, tensors and matrices
1.3.2. Tensors
1.3.3. Transformation of tensors
1.3.3.1. Tensors of the zero-th order – scalars
1.3.3.2. Tensors of the first order – vectors
1.3.3.3. Tensors of the second order
1.3.3.4. Tensors of the fourth order
1.3.4. Stress tensor
1.3.5. Voigt's representation of stress
1.3.6. Strain tensor
1.3.7. Principal axes and invariants of the second order tensor
1.3.8. Examples
1.3.9. Implementations of basic matrix and tensor operations

02_KI. Kinematics

2.1. Deformation and strain
2.2. Material and spatial coordinates – configuration
2.3. Lagrangian and Eulerian formulations of deformation
2.4. Deformation and strain
2.5. Green-Lagrange strain tensor
2.6. Almansi strain tensor
2.7. Cauchy strain tensor – infinitesimal displacements and strains
2.8. Comparison of strain tensors

03_ST. Stress

3.1. What is stress
3.2. Body and traction forces
3.3. True stress and engineering stress
3.4. Motivation for inventing additional stress measures
3.5. The first Piola-Kirchhoff stress tensor
3.6. The second Piola-Kirchhoff stress tensor
3.7. Piola-Kirchhoff stress tensors in the linear theory of elasticity

04_CR. Constitutive relations

4.1. Material models
4.2. Linear elastic material
4.3. Uniaxial stress
4.3.1. Prelude
4.4. Plane stress and plane strain
4.5. Transformation of stress components into another coordinate system
4.6. Mohr's circle representation for the two-dimensional state of stress
4.7. Drawing the Mohr's circle with rule and compasses
4.8. Finding principal stresses by means of the Mohr's circle
4.9. Finding principal stresses by means of the standard eigenvalue problem
4.10. Principal stress
4.11. Plane state of strain
4.12. Transformation of strain components into another coordinate system
4.13. Tension (tensile) testing
4.14. Plasticity
4.15. Axial (longitudinal) vs. radial (transversal, lateral) deformations
4.16. Hooke's law appearances for different types of loading
4.16.1. Hooke's law for 1D
4.16.2. Hooke's law for the plane stress
4.16.3. Hooke's law for 3D
4.16.4. Plane stress vs. plane strain
4.17. Principal stresses – once more

05_SE. Strain energy

5.1 Introduction
5.2. Strain energy for uniaxial stress
5.3. Strain energy for plane state of stress
5.3.1. The strain energy due to normal stress
5.3.2. The strain energy due to shear stress
5.4. Strain energy for the 3D state of stress
5.5. Strain energy in a beam subjected to pure bending
5.6. Strain energy expressed in tensor notation
5.7. Analogy of relations for tension, bending and torsion

06_FT. Failure theories

6.0. Introduction
6.1. Rankine's hypothesis of the maximum stress
6.2. Saint-Venant's hypothesis for the maximum shear
6.3. Guest's hypothesis of the maximum shear stress
6.4. Beltrami's hypothesis of the strain energy density
6.5. Mises's hypothesis of the deviatoric part of the strain energy density
6.5.1. A part of the strain energy density evoked by the volumetric stress
6.5.2. A part of the strain energy density evoked by the deviatoric stress
6.6. Plasticity conditions

07_PS. Principle of superposition

7.1. Introduction
7.2. Betti's theorem
7.3. Maxwell's theorem

08_PV. Principle of virtual work

8.1. Introduction
8.2. Virtual work
8.3. Castigliano's theorems
8.4. Saint-Venant's principle

09_LM. Typical loading modes

9.1. Introduction to bending, torsion and buckling

9.2. **Bending**

9.2.1. Introduction
9.2.1.1. Terminology
9.2.1.2. Explanation of terms: degrees of freedom, static determinacy, etc.
9.2.2. Internal actions in a cross section
9.2.3. Sign conventions
9.2.4. Deformations and stresses in beams subjected to pure bending
9.2.5. The conditions for the safe applicability of the slender beam theory
9.2.6. The second moments of area
9.2.7. Parallel axis-theorem – the second moments of area with respect to shifted axes
9.2.8. The second moments of area with respect to the rotated axes
9.2.9. The influence of the shear force on the deformation of the beam
9.2.10. The differential equation of the deflection curve
9.2.11. Beam examples
9.2.12. Strain energy in a beam subjected to pure bending

9.3. **Torsion**

9.3.1. Introduction
9.3.2. Deformation, strain and stress
9.3.3. How the applied torque (moment) is related to the shear stress in the bar
9.3.4. Analogy of relations for tension, bending and torsion

9.4. **Buckling**

9.4.1. Introduction
9.4.2. Stability
9.4.3. Buckling of slender bars
9.4.4. Four modes of buckling

10_EX. Simple examples

10.1. Tension – compression
10.2. The strain energy and the work exerted by an external force
10.3. Statically indeterminate cases
10.4. Thermal stress

11_FE. A brief survey of finite element method

11.0. Introduction
11.1. Discretization of continuous quantities in continuum mechanics
11.2. Basic equations of continuum mechanics
11.3. Cauchy equations of motion
11.4. Kinematic relations
11.5. Constitutive equations
11.6. Numerical approaches
11.7. The finite element method
11.8. Linear case
11.9. Determination of A, B operators
11.10. Material non-linearity only
11.11. Material and geometrical nonlinearity
11.12. Finite element method in mechanics of deformable bodies
11.13. Rod (bar) element
11.14. Planar beam element
11.15. Planar triangular element with 6 dof's
11.16. Quadrilateral element with 8 dof's
11.17. Coordinate transformation
11.17.1. Rod element (2 dof's) in plane
11.17.2. The planar beam element with 6 dof's
11.18. Assembling
11.19. Assembling algorithm
11.20. Respecting boundary conditions

12_RE. References

01_IN. Introductory part

1.1. Introduction

Studying the subject of mechanics of deformable bodies (also called mechanics of materials or strength of material) we will rely on knowledge obtained during the freshman courses of engineering, namely mathematics (vector, matrix and tensor analyses, and differential and integral calculus), mechanics of rigid bodies (statics, kinematics, dynamics) and the basic principles of mechanical engineering.

1.1.1. A few words about modeling

Since time immemorial people are trying to find out, analyze, explain and predict the phenomena occurring in Nature. At first sight, these phenomena are not evident, they are complicated – it is difficult to understand and analyze them. The motivation for this activity is to understand and thus to gain the ability to predict.

He who knows and can predict is then able to make correct decisions. Throughout ages, such a person is always highly respected in society. Recall tribe shamans, Egyptians priests, managers and last but not least engineers. They know how to treat local maladies and ailments, they know how the rise of the brightest star in the Northern hemisphere – Sirius – is related to the flood of the Nile river and to the consequent harvest, how to send a man to the Moon and back, and how to design an bridge being able to withstand the predictable load.

To find out at least the partial explanations solutions of phenomena Nature, we try to simplify them, neglect seemingly marginal facts, with a pious hope that the neglected parts do not substantially influence the properties of the studied subject. This way, we get a simplified solution, which does not fully describe the original phenomenon. Such a process is called modeling and the result of such a mental process is called the *model*. So, each model inherently contains certain assumptions and simplifications and its validity is thus limited. A model can be expected to be reliable if it is used within the scope of its accepted assumptions. And of course, the model reliability has to be thoroughly tested.

In this text, we will limit interests to the solid *continuum mechanics* and to its subset, i.e. to the engineering *strength of material*, also called the *mechanics of materials*.

1.1.2. Continuum mechanics

Continuum mechanics is a model dealing with the response of solid or fluid media to external influences.

Continuum mechanics analyses the response of solid and fluid media to external effects. By the term response, we understand the spatial and temporal distributions of displacements, velocities, accelerations, forces, stresses, and strains, etc., associated with individual particles of the medium. The external effects could represent the force loadings, the thermal loadings, the prescribed deformations, etc.

The continuum is considered one of the possible macroscopic models of the matter. The continuity itself is a property closely dependent on the magnification scale being used for the observation of analyzed specimens. We have to realize that the matter in Nature is actually corpuscular and thus not continuous.

Accepting the continuum model, we intentionally neglect the corpuscular nature of matter; we assume that the matter is continuously distributed within the body. We claim that all the material properties of an infinitesimal element are identical with those of a specimen of the finite size. The quantities describing the response of the body are assumed to be continuous functions of space and time.

In *fluids*, the molecules are allowed to move relatively freely, being constrained by weak intermolecular forces, while in gases the intermolecular forces are still weaker and the particle motions are rather unlimited.

So, the *solid continuum mechanics* – which is the subject treated in this text – is a model of Nature being characterized by the fact that within the examined solid bodies the relative motions of material particles are limited by strong inter-atomic forces.

The equations describing the behaviour of the solid continuum model are based on kinematics and on the basic physical laws related to the conservation of energy, momentum, and energy.

Due to the accepted assumptions mentioned above, the continuum model has a limited scope of validity. What are those limits cannot be mathematically derived and expressed – it is always the properly conceived experiment which certifies the theory.

Within the scope of solid continuum mechanics, we will deal with deformable solids for which there are strong inter-atomic forces allowing solid particles limited displacements only. The solid continuum model is considered reliable if the size of the critical analyzed element of the matter is at least 10^4 times greater then the inter-atomic distance of the material the body is made of. This empirical wisdom comes from [19]. For metals the inter-atomic distance is about 10^{-10} m, so the critical element size should not be less than that of 10^{-6} m.

This critical size also limits the maximum frequency that can be safely transferred by the solid continuum model. Imagine a harmonic wave whose wavelength $\lambda = 10^{-6}$ m is equal to the critical element size mentioned above. It is known that a stress wave in steel materials propagates with the velocity of about $c = 5000$ m/s. From it follows that the maximum frequency that could be reliably modeled by continuum is $f_{\max} = c / \lambda \approx 5$ GHz .

This value is sufficiently high above the frequencies currently occurring in mechanical engineering practice and justifies the safe usage of the continuum model even for stress wave propagation phenomena.

See [18], [23].

1.1.3. Strength of material

The subject of the strength of material, as it is taught in engineering curricula, is a subset of solid continuum mechanics. It deals with ascertaining deformation, strains, and stresses in deformable bodies (design elements of machines, structures) due to external loadings. Also, a prediction – related to the ability to withstand the prescribed loading – is studied. The subject of the strength of material is also related to dynamical problems allowing to analyze the impact problems with stress propagation phenomena. Then, the loadings and consequent deformations, strains and stresses are not only functions of space but also functions of time.

See [21], [40].

1.1.4. Linear vs. non-linear

The linear solid continuum mechanics is based on the following assumptions.

Infinitesimal strains
For linear cases, it is characteristic that the strain is expressed as the first derivative of displacements with respect to un-deformed coordinates of the examined body. Derivatives of the higher order are neglected.

Small displacements
It is assumed that maximum displacements of the deformed body are small with respect to the overall dimensions of the considered body. It is tacitly assumed that under the term of small displacements we understand both displacements and rotations.

Equilibrium equations are written with respect to the initial, un-deformed configuration. It means that deformations and strains due to the prescribed loadings are properly evaluated, but resulting forces and stresses are computed from the geometry of the initial, un-deformed configuration of the body. It comes from the previously stated assumptions of the small overall deformations.

Linear constitutive relation
The validity of Hooke's law, supposing that there is a linear relation between stress and strain quantities, is assumed. Theoretically, there is no limit of this linear behaviour, thus the processes of plasticity, hysteresis, and permanent material damage are not considered.

Boundary conditions do not change due to the loading. It is assumed that the boundary conditions do not change during the loading process.

1.1.5. Sources of non-linearity

Generally, the world is non-linear. To simplify the modeling process, it is worthwhile to classify the individual sources of non-linearity and to use only those that are pertinent to the particular engineering case being analyzed.

Material non-linearity only
Non-linear material models, as plasticity, viscoelasticity, creep, etc., are usually combined with assumptions of small strains and small displacements.

Large displacements, small strains
This type of material non-linearity is relatively common in engineering practice. As an example, the behaviour of flexible truss and shell structures can be mentioned. In this case, the large displacements of structures are combined with the local linear behaviour of the material. So, the Hooke's law is locally valid. What happens to a material element during its deformation is depicted in Fig. IN_1.

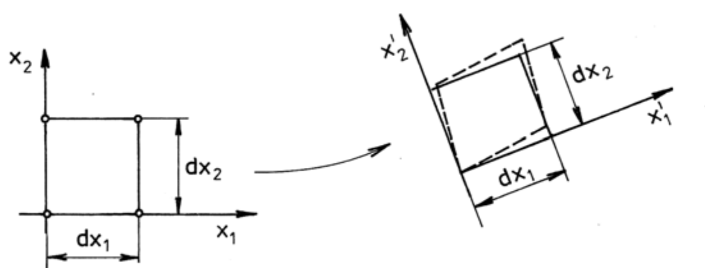


Fig. IN_1 … Large disp small strains

Large displacements, large strains, non-linear material behaviour
This is the generic case, which is most difficult to solve. Often, the boundary conditions might be changed during the loading process. Examples: contact problems, post-buckling behaviour of structures, technological processes with material forming, etc. See Fig. IN_2.

Fig. IN_2 … Large disp large strains

See [4], [7].

Differences between rigid and deformable mechanics

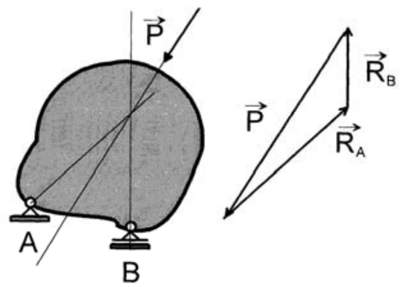


Fig. IN_3 … Rigid body

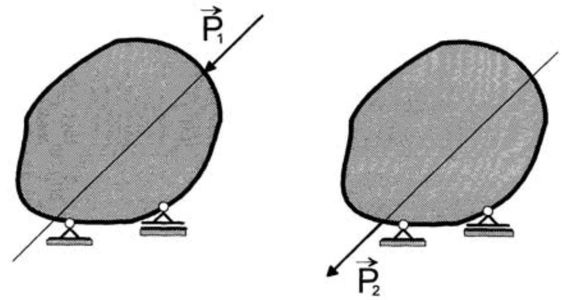


Fig. IN_4 …Deformable body

In rigid body mechanics, see Fig. IN_3, the state of equilibrium of applied loads with reactions forces can be analytically found only for the statically determinate cases. The applied force is freely movable along its line of action having thus no effects on reactions.

In mechanics of deformable bodies, see Fig. IN_4, the cases with statically indetermined conditions could be solved as well, but the equilibrium conditions have to be accompanied by a suitable number of deformable conditions. Furthermore, the acting force cannot be freely moved along its line of action and thus the force P_1, in Fig. IN_4, causes a different stress and strain distributions in the loaded body than the force P_2.

1.1.6. System of units

In this text, we will systematically use quantities expressed in units defined in The International System of Units, universally abbreviated SI (from the French Le Système International d'Unite´s). See [37].

Seven base SI quantities and their units are

Base quantity	name	symbol
length	meter	m
mass	kilogram	kg
time	second	s
electric current	amper	A
thermodynamic temperature	kelvin	K
amount of substance	mole	mol
luminous intensity	candela	cd

Some of SI derived units used in mechanics

Derived quantity	special name	special symbol	in base units
area	square meter		m^2
volume	cubic meter		m^3
speed, velocity	meter per second		m/s
acceleration	meter per second squared		m/s^2
wave number	reciprocal meter		m^{-1}
density	kilogram per cubic meter		kg/m^3
frequency	hertz	Hz	s^{-1}
force	newton	N	$kgm\,s^{-2}$
pressure, stress	pascal	Pa	$Nm^{-2} = kgm^{-1}s^{-2}$
energy, work	joule	J	$Nm = kgm^2s^{-2}$
power	watt	W	$Js^{-1} = kgm^2s^{-3}$

A note to weight and mass

In science and technology, the weight of a body is defined as the force that gives the body the acceleration equal to the local gravitational acceleration, while the mass is a measure of matter determining the aversion of a body to move with acceleration. Thus, the SI unit of the quantity called weight, defined in this way, is newton [N]. However, in everyday use, and among the laic community, the term weight is frequently but wrongly, used as a synonym for mass. So, highly questionable are the common vocabulary entries claiming that the mass of a body is determined by weighing. This is not true – one kilogram of gold is heavier on the Pole than on the Equator of the Earth. Nevertheless, the above heretic statements would never be used in this text.

Old fashioned and 'unacceptable' units

Technical system of units

There are many units that are outside the SI system that are not formally accepted but are still often used. The so-called technical system of units takes as the base quantities the length, the force and the time – they are measured in meters [m], kiloponds, denoted [kp] or [kg*], and seconds [s], respectively. One kilopond [kp] is defined as the weight of a body having the mass of one kilogram [kg]. The mass unit in this system is [$1\,kps^2/m$].

Since the weight G of a body, having the mass m, is the force induced by the local gravitational acceleration g, then using Newton's law we get the relation between weight and mass in the form $G = mg$. For the standard gravitational acceleration, we get $1\,kp = 9.8061\,N$.

In certain respects, this system of units is more ‘human-oriented’ than the SI system. For example, a body having the mass of 1 kg weighs just 1 kp, the pressure in the depth of 10 meters of water is 1 atmosphere or $1\ \mathrm{kp/m^2}$, etc. From the point of view of the plain common sense, this approach was very convenient and was easily grasped, but the fact that the weight depends on the local value of gravitational acceleration made this system physically unacceptable.

Imperial system of units

In the United States, they are still using a version of the technical system of units, expressed, however, in imperial units, i.e. pound_force, foot, second. The term pound_force, [lb_force], is used as a unit of weight, while for the mass they have pound_mass denoted [lb_mass] or [poundal] or slug. Its unit is $[1\,\mathrm{lb\_force}\ \mathrm{s}^2/\mathrm{ft}]$.

1.2. History of mechanics of rigid and deformable bodies

History of mechanics of rigid and deformable bodies goes back to Galileo (1564 – 1642) who analyzed the deformations and mechanical failures of rods, beams and hollow cylinders due to external loadings. See Fig. IN_5 and [16].

(From the *Discorsi*, Leiden 1638.)

Fig. IN_5 …Galileo beam

Robert Hook (1635 – 1702) is the founder of the modern concept of the theory of elasticity. In his contribution De potenziâ restitutiva, published in 1678, he claims that he invented the theory of springs. The term spring in his interpretation is to be understood not only as the spiral or leaf spring but also as the ‘springing body’. His famous statement, which in Latin is *Ut tensio sic vis*, is translated into English as *The power of any spring is in the same proportion with the tension thereof*. This might be reformulated as *the elongation of the spring is proportional to the force*. Today’s formulation of Hooke’s law is – *the stress* σ *is proportional to the strain* ε.

Thomas Young (1773 – 1829) was the person with a wide range of interests covering medicine, languages, and mechanics. The coefficient of proportionality E, appearing in Hook’s law, i.e. $\sigma = E\varepsilon$, is named after him.

Claude-Louis Navier (1785 – 1836) was a French engineer and physicist who also specialized in mechanics. For the first time, he formulated equations of motion for a generic particle of a loaded body.
Augustin-Louis Cauchy (1789 – 1857) made substantial contributions to the analysis of solid continuum mechanics. He accepted the stress definition established by Saint-

Venant[1] (1797 – 1886), defined the stress ellipsoid, the principal stress, derived the equations describing the equilibrium of forces acting on an infinitesimal element and published, what we might today call, the generalized Hooke's law – expressing thus the linear relations between stress and strain components of a loaded body in 3D space. He is responsible for the fact that the stress tensor is considered symmetric[2].

An excellent source of information concerning the history of elasticity, the history of the strength of material is provided in [38]. See also [40].

It reveals, that the analysis of the response of deformable bodies to external loadings evolved historically by two independent ways – an engineering and mathematical.
The engineering approach was based on consequent and rather independent analyses of bodies of specific forms as rods, strings, beams, shells, vessels, etc. being subjected to different types of external loads as the force, moment, pressure, etc.

The mathematical attitude started by a generic formulation of equilibrium conditions, or equations of motion, for an infinitesimal element of a particular body with the intention to determine the distribution of displacements, velocities, accelerations, strains, stresses in space and time. This process leads to partial differential equations, which are to be solved for the prescribed boundary and initial conditions – not an easy task.

So, side by side there are two approaches leading to two different educational styles that are supported by historical evolution. Namely, the subject of the engineering strength of material, and the mathematically oriented theory of elasticity – more generally the continuum mechanics theory.

The former, represents the bottom-to-top approach, starting with the analyses of simple cases of geometry for different kinds of loadings and gradually proceeding to the complicated ones. It represents the substance of engineering approach to the problem solving – always trying to find out a simplified, but within accepted assumptions 'correct' solutions, minimizing the required effort to do so, and using for this purpose the available computational tools. Our forefathers did not have computers at their disposals.

The latter, top-to-bottom approach went the opposite way. Until recently, the mathematical theory of elasticity was considered to be a purely academic matter, since the resulting partial differential equations, describing the time and space distributions of kinematical and stress quantities of loaded bodies, applied to generic initial and boundary conditions, did not as a rule have close analytical solutions. Thus, the direct application of the mathematical theory of elasticity to engineering problems was initially almost negligible. However, the rise of computers in the middle of the last century, accompanied by efficient implementations of numerical methods, led to the renaissance of the

[1] His full name is Adhémar Jean Claude Barré de Saint-Venant.

[2] Cosserat brothers (François and Eugéne), in *Théorie des corps déformables (Theory of deformable bodies)* (1909), established an alternative theory of elasticity in which the stress tensor is not symmetric.

mathematical theory of elasticity and allowed the sudden rise of effective tools as the finite element method, the boundary element method, etc.

Both approaches will be presented in this text – the comprehension of the mathematical theory of elasticity allows better understanding of the theoretical backgrounds of modern computational tools while the knowledge of principles of the engineering strength of material permits to solve simple cases off-hand, to have a proper feelings for the ability of basic design parts to withstand the applied loading, and last but not least to have a computing etalons and benchmarks for checking the first approximations of solutions of complicated cases in engineering practice.

There are two relatively distinct mathematical tools that are suited to the above-mentioned approaches. The mathematically oriented continuum mechanics theory is efficiently described and analyzed by tensors, while for the engineering approach and for the consequent programming efforts it is the matrix description which is preferable. We will show that after all both the tools are closely related and interwoven. Both approaches are useful for the proper understanding of modern engineering tools, as the finite element method, boundary element method, etc., that are primarily used for analyzing the state of stress in machine parts and the ability of those parts to withstand the applied loading.

1.3. Mathematical and computational tools – background

1.3.1. Scalars, vectors, tensors and matrices

The quantities we are dealing with in the continuum mechanics (as displacements, forces, stresses, etc.) are as rule independent of the coordinate system in which they are expressed. The quantities of that type are suitably represented by vectors and tensors, for their elegance, shorthand brevity and contextual richness.

What is the meaning of the independence of vector and tensor quantities with respect to a particular coordinate system?

Take for example the *vector*, which in mechanics could represent the displacement, velocity, acceleration, force, etc. We are frequently visualizing it as an arrow, being defined by its orientation and length. Often, we are working with its components that are actually the projections of that vector into three mutually perpendicular axes – the Cartesian coordinate axes.

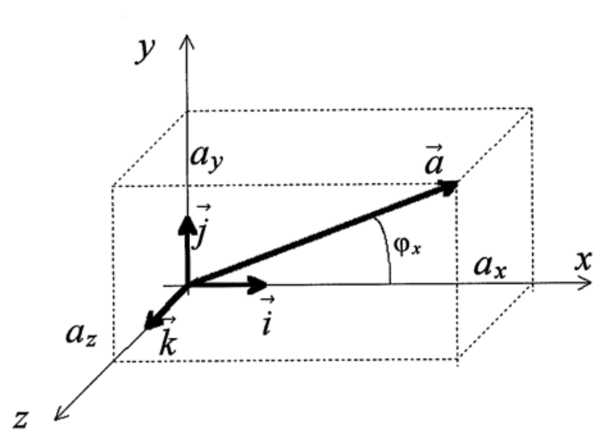


Fig. IN_6 … Vector cartesian

Thus, a vector in 3D space has generally three independent projections – components. See Fig. IN_6.

We could, however, define infinitely many independent coordinate systems. While the considered vector is still the same, its components – the vector components – are different, depending on that former choice. We say that the vector is invariant with respect to a particular choice of the coordinate system. We will show that there is a unique procedure, allowing expressing the vector components of the same vector from one coordinate system to another.

1.3.2. Tensors

Similarly, in continuum mechanics, an entity called *tensor* could suitably represent the state of stress in a particular particle of a body. The state of stress is a quantity of tensor nature – in 3D space, it has 9 components and in 2D space, there are four components. As before, while the stress tensor is independent of the choice of the coordinate system, its components – the stress components – differ, depending on the choice of the particular coordinate system. The different stress components (simply called stresses) of the same tensor could be easily expressed in different coordinate systems but the stress tensor, signifying the state of stress, is still the same. Similarly, for the strain quantities.

Here, we briefly explain the working tools and operators suitable allowing an efficient treatment of quantities appearing in continuum mechanics.

The tensor is a mathematical entity uniquely defined by the relations prescribing transformation of its components from one coordinate system to another. In continuum mechanics, the tensors will be mainly used for the representation of stress and strain quantities. In this paragraph, we will concentrate on their mathematical properties.

References to textbooks related to tensor and matrix analyses are numerous. See for example [33], [28], [14], [15], [24], [25], [35], [36].

1.3.3. Transformation of tensors

Generally, the tensors are quantities uniquely defined by the prescription of their transformation properties.

In this text we will limit our attention to tensors living in Cartesian coordinate systems[3] – we call them the Cartesian tensors. Tensors are classified by their *order*, sometimes called *rank*. The number of their components depends on their 'spatiality'. For example, in a 3D space, i.e. for $n = 3$, the tensor of the $N-\text{th}$ order has n^N components. In this text, the quantity n, denoting the 'spatiality' will be examined for values 1, 2 or 3, while the quantity N, determining the order of tensor, will reach values from 0 to 4 only.

[3] The Cartesian coordinate system is represented by mutually perpendicular axes. Any 3D vector can be expressed as a linear combination of three non-coplanar vectors – called the base vectors.

1.3.3.1. Tensors of the zeroth order – scalars, $N = 0$

The scalars are quantities uniquely determined by their magnitudes. They will be denoted by Latin or Greek letters and printed in italics. Examples are the temperature, say T, the mechanical work W, the density ρ, etc. Scalars do not change their values when expressed in different coordinate systems.

1.3.3.2. Tensors of the first order – vectors, $N = 1$

The vectors, in this context, are subsets of tensors – the tensors of the first order. They are characterized by a single free index and uniquely determined by their orientation and magnitude. Examples are displacement, velocity, acceleration, force, etc. They will be denoted either by the bold straight fonts or by the italics font accompanied by an overhead arrow. As an example, take the velocity vector, which might be denoted as $\mathbf{v}$ or $\vec{v}$ or v_i.

The vector components are usually collected in braces $-\{\ \}$. For example, for a radius vector of a particle with coordinates $x_1\, x_2\, x_3$ we might write $\vec{x} = \mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$. In this case, the column vector was used, sometimes we work with row vectors, as $\{x_1\ x_2\ x_3\}$. In tensor analysis, it is convenient to name the components as x_1, x_2, x_3 instead of x, y, z, since it allows an efficient dealing with quantities appearing in formulas.

For example, the length of the radius vector $\mathbf{x}$, i.e. the scalar quantity r, could be expressed by means of the Pythagoras theorem as $r = |\vec{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{x_1 x_1 + x_2 x_2 + x_3 x_3}$. Then, square of that length is $r^2 = \sum_{i=1}^{3} x_i x_i$. This expression could be even more simplified, by using the so-called *Einstein summation convention*, by writing $r^2 = x_i x_i$. The rule states that in case of repeated indices the summation sign might be omitted.

Primarily, the vector in this text is formally considered as the column quantity[4]. To save the printing space, we might express the column vector as a row one, using the transpose operator known from the matrix analysis. For example, the velocity vector might be expressed as $\vec{v} = \mathbf{v} = \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \{v_1 \quad v_2 \quad v_3\}^{\mathrm{T}}$. Again, instead of v_x, v_y, v_z, it is preferable to write v_1, v_2, v_3.

[4] In Matlab, by default, the vector quantities are considered as row arrays.

Transformation of a tensor of the first order – i.e. the vector

To simplify the explanation, let's start with a 2D space example where there are two Cartesian coordinate systems, having the same origin but a different angular orientation – the primed and unprimed coordinate systems. In that Cartesian space lives a vector $\vec{a}$. See Fig. IN_7 where its projections into axes of both coordinate systems are depicted. The relation (also called the transformation) between components of the same vector in two different coordinate systems, is obtained by mere inspection

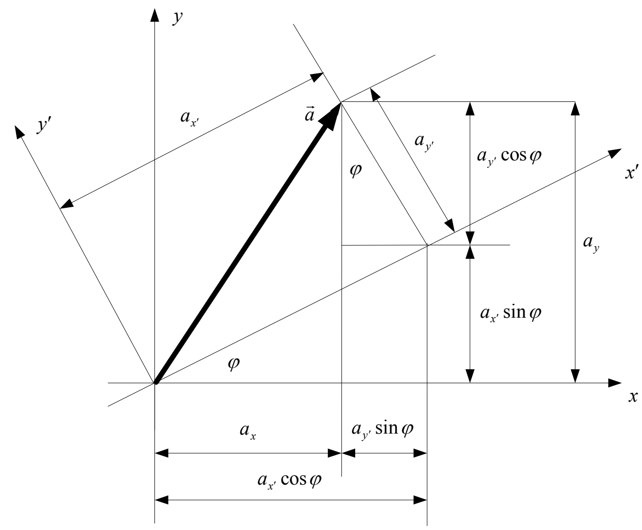


Fig. IN_7 … Vector components in two coordinate systems

$$\begin{aligned} a_x &= a_{x'}\cos\varphi - a_{y'}\sin\varphi, \\ a_y &= a_{x'}\sin\varphi + a_{y'}\cos\varphi. \end{aligned} \qquad \text{(IN\_1)}$$

This relation, written in the matrix form, gives

$$\begin{Bmatrix} a_x \\ a_y \end{Bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{Bmatrix} a_{x'} \\ a_{y'} \end{Bmatrix}; \qquad \mathbf{a} = \mathbf{R}\mathbf{a}'. \qquad \text{(IN\_2)}$$

In this case, the transformation matrix $\mathbf{R}$ represents the rotation process and is said to be orthogonal. Such a transformation conserves the lengths of vectors; geometrically it represents the rotation or the mirroring. For an orthogonal matrix, its determinant $\det\mathbf{R} = \pm 1$. The inverse of such a matrix is obtained by a mere transposition, i.e. $\mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}}$. So, in this case, the inverse transformation is defined by

$$\begin{Bmatrix} a_{x'} \\ a_{y'} \end{Bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \begin{Bmatrix} a_x \\ a_y \end{Bmatrix}; \qquad \mathbf{a}' = \mathbf{R}^{\mathrm{T}}\mathbf{a}. \qquad \text{(IN\_3)}$$

Denoting the indices by integers 1,2 instead of letters x, y allows using a simple and elegant notation in the form

$$a'_i = R_{ji} a_j. \qquad \text{(IN\_4)}$$

Notice, that the index j, appearing twice on the right-hand side, is thus the summation index. The index i, called the *free index,* is understood to take all the possible values from 1 to n, which in this 2D example is 2. The summation index is called the dummy index, since the letter j could be replaced by any imaginable letter (say $k, l, m,$ etc.), not being in 'conflict of interests' with the free index – that is i in this case. So, the previous formula actually represents two equations, both containing the summation. They have the form

$$a'_i = \sum_{j=1}^{2} R_{ji} a_j \quad \text{for} \quad i = 1,2. \tag{IN\_5}$$

This relation, however, represents the algorithm for obtaining the result of the matrix by column vector multiplication known from the matrix algebra. Explicitly, written in full, we have two equations

$$\begin{Bmatrix} a'_1 \\ a'_2 \end{Bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{bmatrix} a_1 \cos\varphi + a_2 \sin\varphi \\ -a_1 \sin\varphi + a_2 \cos\varphi \end{bmatrix} \quad \text{or} \quad \begin{aligned} a'_1 &= a_1 \cos\varphi + a_2 \sin\varphi \\ a'_2 &= -a_1 \sin\varphi + a_2 \cos\varphi \end{aligned}.$$

… (IN_6)

In this case, the inverse transformation is defined simply by

$$\begin{Bmatrix} a'_1 \\ a'_2 \end{Bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}; \qquad \mathbf{a}' = \mathbf{R}^{\mathrm{T}} \mathbf{a} \qquad \text{or} \qquad a'_i = R_{ji} a_j. \tag{IN\_7}$$

This way, we have shown the convenience of representing the component counters by numbers instead of letters and also the close connection of tensor and matrix representations.

Vector transformation in 3D

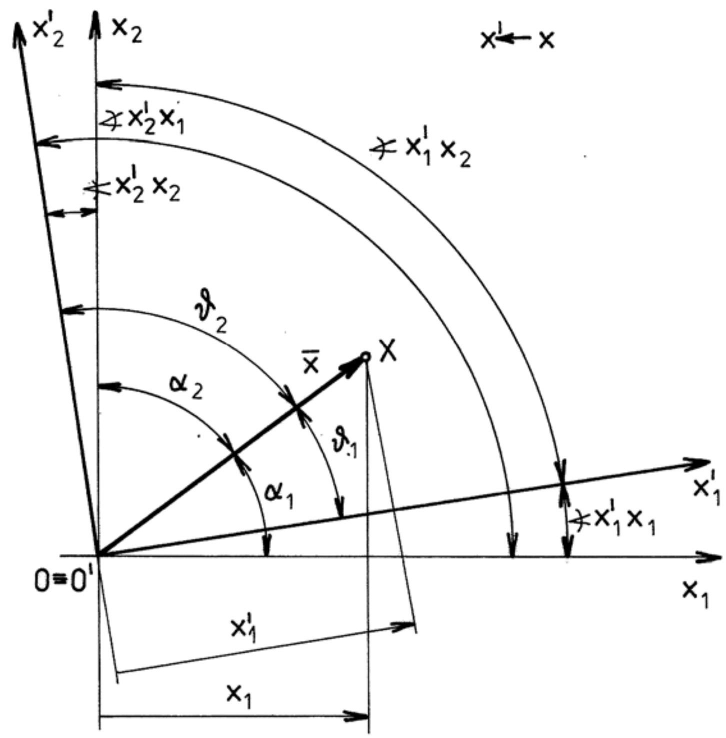


Fig. IN_8 … 3D coor transf

In Fig. IN_8 there is a generic vector $\mathbf{x}$. Let the axes $O\,x_1, x_2, x_3$ and $O'\,x_1'\,x_2'\,x_3'$ represent two right-handed Cartesian coordinate systems with a common origin at an arbitrary point $O \equiv O'$. For simplicity, a 2D sketch is plotted only.

If a symbol R_{ij} represents the cosine of an angle between *i*-th primed and *j*-th unprimed coordinate axes i.e. $R_{ij} = \cos(\text{angle between}\, x_i' x_j) = \cos(\angle x_i' x_j)$, then all the nine components can be arranged into a 3×3 matrix $\mathbf{R} = [R_{ij}]$, that is called the *rotation matrix* or the *transformation matrix*, or the *matrix of direction cosines*. Then, the transformation of components of a generic vector $\mathbf{x}$ from the non-primed to the primed coordinate system is provided by the formally same formula as before, i.e. by $\mathbf{x} = \mathbf{R}\mathbf{x}'$ or $x_i = R_{ij} x_j'$. In 3D, this formula is understood as

$$x_i = \sum_{j=1}^{3} R_{ij} x_j' \quad \text{for} \quad i = 1,2,3\,. \tag{IN\_8}$$

The formula represents three equations – in each of them, there is a triple summation. Try to write the above formula in full.

Similarly, for the inverse transformation

$$\mathbf{x}' = \mathbf{R}^{\mathrm{T}}\mathbf{x} \qquad \text{or} \qquad x_i' = R_{ji} x_j \,. \tag{IN\_9}$$

In n-dimensional space, the tensors of the first order (i.e. the vectors) have n components.

True vectors versus one-dimensional arrays having *m* components

It should be emphasized that we have to distinguish the true vectors[5], being defined as oriented arrows with prescribed magnitudes that are living in 1D, 2D or 3D spaces, for which the above-mentioned transformation property holds, and one-dimensional arrays, defined in programming languages, that could generally contain m components where m could be any (finite and positive) integer. In texts dedicated to programming, these arrays are often called vectors as well. This might cause a sort of confusion because both mentioned 'vectors' have to be treated differently when being transferred from one coordinate system to another.

1.3.3.3. Tensors of the second order, $N = 2$

The tensor of the second order, characterized by two free indices, is defined as a dyadic product of two column vectors, say, $\mathbf{a} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$, $\mathbf{b} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$ and can be formally written, denoted and expressed by different ways as

$$\mathbf{T} = \mathbf{a} \otimes \mathbf{b}, T_{ij} = a_i b_j, \mathbf{T} = \mathbf{a}\mathbf{b}^{\mathrm{T}} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \begin{Bmatrix} b_1 & b_2 & b_3 \end{Bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

… (IN_10)

In this text, these tensors are denoted by bold straight capital letters as $\mathbf{T}$, while their components by italics accompanied by two lower right indices, as T_{ij}. The notation T_{ij} might be understood by two distinct but complementary meanings. Either as the tensor component for particular values of i and j, or as the 'whole' tensor defined for all the range of applicable indices.

The detailed derivation of these relations is in [28].

Orthogonal transformation of the second order tensor between two Cartesian coordinate systems having the same origin but a mutually different angular orientation can be expressed by

---

[5] Generally, the attribute 'true' is not accentuated.

$$T'_{ij} = R_{ik} T_{kl} R_{jl} \quad \text{or} \quad \mathbf{T}' = \mathbf{R}\mathbf{T}\mathbf{R}^{\mathrm{T}}\,, \tag{IN\_11}$$

where $R_{ij} = \cos(\text{angle between } x'_i x_j)$. The matrix $\mathbf{R}$ is often called the matrix of direction cosines.

The inverse transformation is

$$T_{ij} = R_{ki} T'_{kl} R_{lj} \quad \text{or} \quad \mathbf{T} = \mathbf{R}^{\mathrm{T}}\mathbf{T}'\,\mathbf{R}\,. \tag{IN\_12}$$

1.3.3.4. Tensors of the fourth order, $N = 4$

The forward and inverse transformation laws for tensors of the fourth order can be expressed as

$$C'_{ijkl} = A_{ir} A_{js} A_{kt} A_{ln} C_{rstn} \quad \text{and} \qquad C_{ijkl} = A_{ri} A_{sj} A_{tk} A_{nl} C'_{rstn}\,. \tag{IN\_13}$$

We have stated that the tensor expressions are compact tools allowing effective description of quantities characterizing the response of solid bodies to external loadings. These simply appearing expressions actually require a lot of work to do if there is a necessity to dirty our hands with its evaluation.

In 3D space this kind of tensor has $n^N = 3^4 = 81$ elements and in continuum mechanics is suitable for expressing the components of coefficients of Young modulus appearing in the generalized Hooke's law which has the form

$$\Sigma_{ij} = C_{ijkl} E_{kl}\,. \tag{IN\_14}$$

In matrix algebra, there is no direct equivalent for tensor quantities of the fourth order.

Evaluation of the first formula shown above requires implementing four cycles to address the indices i, j, k, l and an additional four cycles for quadruple summation indicated by indices r, s, t, n. We could simply proceed as shown in Matlab program 3.

```
% Matlab program 3
% fourth_order_tensor_transformation for n = 3
for i = 1:n
  for j = 1:n
    for k = 1:n
      for l = 1:n
        C_prime(i,j,k,l) = 0;
        for r = 1:n
          for s = 1:n
            for t = 1:n
              for u = 1:n
       C_prime(i,j,k,l) = C_prime(i,j,k,l) + A(i,r)*A(j,s)*A(k,t)*A(l,u)*C(r,s,t,u);
              end
            end
          end
        end
      end
```

```
    end
  end
end
```

Instead of C' we write `C_prime` in the program.

1.3.4. Stress tensor

The stress tensor, say $\mathbf{\Sigma}$, is a **symmetric** tensor of the second order. In 3D space it has 9 components – say Σ_{ij}, $i = 1$ to 3, $j = 1$ to 3 – they can be assembled into a 3×3 matrix[6] as follows

$$\mathbf{\Sigma} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{bmatrix}. \qquad \text{(IN\_16)}$$

The physical meanings of the stress tensor components (sometimes simply called stresses) are presented in the paragraph devoted to stress, i.e. 03_ST. Stress.

This is the way, how the stress components are expressed in the mathematical theory of elasticity. The symmetry of the stress tensor means that $\Sigma_{12} = \Sigma_{21}, \Sigma_{13} = \Sigma_{31}, \Sigma_{23} = \Sigma_{32}$, so there are actually only six independent stress components out of nine.

1.3.5. Voigt's representation of stress

This fact historically led to the engineering notation of stress that works only with six independent components – they are usually assembled into a column array as follows

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{Bmatrix} \Sigma_{11} \\ \Sigma_{22} \\ \Sigma_{33} \\ \Sigma_{12} \\ \Sigma_{23} \\ \Sigma_{31} \end{Bmatrix} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}. \qquad \text{(IN\_17)}$$

This way of assembling the stress components, efficiently employing the tensor symmetry, typical for the engineering concept of the strength of material, is known as the Voigt's[7] notation.

---

[6] In 2D space, the corresponding matrix is 2×2. In 1D space, there is one stress component only, so the corresponding 1×1 matrix degenerates into a scalar quantity.

[7] Woldemar Voigt (1850 – 1919), a German physicist. He dealt with crystal physics, thermodynamics, electro optics, mechanics, etc. He was the first who used the term tensor in its today's meaning.

The notation shown in the first column is suitable for programming purposes – it is just a one-dimensional array containing six naturally counted terms, the second column contains unrepeated components of the stress tensor, and the third column contains the same quantities but uses the notation currently used in engineering. Notice, that the first three positions belong to so-called *normal stress components*, the remaining positions serve for the allocation of *shear stress components*. Their order is prescribed by the cyclic combination of indices.

It should be emphasized again that we are dealing with the same physical phenomenon, i.e. the same stress of state, which is, however, expressed by differently assembled and denoted stress components.

The Voigt's stress array (sometimes incorrectly called the stress vector) is not a vector in the proper tensor sense of the word. The transformation, shown above for tensors, in the form

$$\Sigma'_{ij} = R_{ik}\Sigma_{kl}R_{jl} \quad \text{or} \quad \mathbf{\Sigma}' = \mathbf{R}\,\mathbf{\Sigma}\,\mathbf{R}^{\mathrm{T}}, \tag{IN\_18}$$

does not apply to the transformation of the Voigt's stress array $\boldsymbol{\sigma}$. Instead, we have to use a different formula, namely

$$\sigma_i = B_{ij}\sigma_j \quad \text{or} \quad \boldsymbol{\sigma}' = \mathbf{B}\boldsymbol{\sigma} \tag{IN\_19}$$

where

$$\mathbf{B} = \begin{bmatrix} R_{11}^2 & R_{12}^2 & R_{13}^2 & 2R_{11}R_{12} & 2R_{12}R_{13} & 2R_{13}R_{11} \\ R_{21}^2 & R_{22}^2 & R_{23}^2 & 2R_{21}R_{22} & 2R_{22}R_{23} & 2R_{23}R_{21} \\ R_{31}^2 & R_{32}^2 & R_{33}^2 & 2R_{31}R_{32} & 2R_{32}R_{33} & 2R_{33}R_{31} \\ R_{11}R_{21} & R_{12}R_{22} & R_{13}R_{23} & R_{11}R_{22}+R_{21}R_{12} & R_{12}R_{23}+R_{22}R_{13} & R_{13}R_{21}+R_{23}R_{11} \\ R_{21}R_{31} & R_{22}R_{32} & R_{23}R_{33} & R_{21}R_{32}+R_{31}R_{22} & R_{22}R_{33}+R_{32}R_{23} & R_{23}R_{31}+R_{33}R_{21} \\ R_{31}R_{11} & R_{32}R_{12} & R_{33}R_{13} & R_{31}R_{12}+R_{11}R_{32} & R_{32}R_{13}+R_{12}R_{33} & R_{33}R_{11}+R_{13}R_{31} \end{bmatrix}.$$

… (IN_20)

The elements of the $\mathbf{B}$ matrix, composed of functions of elements of the matrix of direction cosines, i.e. $\mathbf{R}$, were obtained by evaluating individual components Σ_{ij} of the tensor formula Eq. (IN_18) . Then, they are assigned to the Voight's stresses σ_i appearing in Eq. (IN_19).

It is a lengthy and rather tiresome procedure, but the Matlab symbolic toolbox program can help and to explains how it might be done.

```
% derive_B_matrix
syms A11 A12 A13 A21 A22 A23 A31 A32 A33 ...
S11 S12 S13 S21 S22 S23 S31 S32 S33 s1 s2 s3 s4 s5 s6 B
```

```
A =  [A11 A12 A13; A21 A22 A23; A31 A32 A33];        % transformation matrix
AT = [A11 A21 A31; A12 A22 A32; A13 A23 A33];        % its transpose
SS = [S11 S12 S13; S21 S22 S23; S31 S32 S33];        % stress tensor in initial config.
SSP = A*SS*AT;                                       % stress tensor in primed config.
SSP = expand(SSP);

% use the Voigt’s notation and
% substitute s1 to s6 in tensor notation
SSP1 = subs(SSP,S11,s1);  SSP1 = subs(SSP1,S22,s2); SSP1 = subs(SSP1,S33,s3);
SSP1 = subs(SSP1,S12,s4); SSP1 = subs(SSP1,S23,s5); SSP1 = subs(SSP1,S31,s6);
SSP1 = subs(SSP1,S21,s4); SSP1 = subs(SSP1,S32,s5); SSP1 = subs(SSP1,S13,s6);

% extract Voigt’s terms from the tensor formulation
ssp1 = SSP1(1,1); ssp2 = SSP1(2,2);          % the first and second terms
ssp3 = SSP1(3,3); ssp4 = SSP1(1,2);          % the third and fourth terms
ssp5 = SSP1(2,3); ssp6 = SSP1(3,1);          % the fifth and sixth terms

% collect terms by s1 to s6
B11 = subs(ssp1,{s1,s2,s3,s4,s5,s6},{1,0,0,0,0,0});
B12 = subs(ssp1,{s1,s2,s3,s4,s5,s6},{0,1,0,0,0,0});
B13 = subs(ssp1,{s1,s2,s3,s4,s5,s6},{0,0,1,0,0,0});
B14 = subs(ssp1,{s1,s2,s3,s4,s5,s6},{0,0,0,1,0,0});
B15 = subs(ssp1,{s1,s2,s3,s4,s5,s6},{0,0,0,0,1,0});
B16 = subs(ssp1,{s1,s2,s3,s4,s5,s6},{0,0,0,0,0,1});

% ...... an obvious part of the program is omitted here

B61 = subs(ssp6,{s1,s2,s3,s4,s5,s6},{1,0,0,0,0,0});
B62 = subs(ssp6,{s1,s2,s3,s4,s5,s6},{0,1,0,0,0,0});
B63 = subs(ssp6,{s1,s2,s3,s4,s5,s6},{0,0,1,0,0,0});
B64 = subs(ssp6,{s1,s2,s3,s4,s5,s6},{0,0,0,1,0,0});
B65 = subs(ssp6,{s1,s2,s3,s4,s5,s6},{0,0,0,0,1,0});
B66 = subs(ssp6,{s1,s2,s3,s4,s5,s6},{0,0,0,0,0,1});

% print the result
B = [B11 B12 B13 B14 B15 B16; ...
B21 B22 B23 B24 B25 B26;      ...
B31 B32 B33 B34 B35 B36;      ...
B41 B42 B43 B44 B45 B46;      ...
B51 B52 B53 B54 B55 B56;      ...
B61 B62 B63 B64 B65 B66 ]

% end of derive_B_matrix
```

1.3.6. Strain tensor

In 3D space, the strain quantity might be represented by the symmetric tensor $\mathbf{E}$ of the second order with nine components, $E_{ij}, i = 1,3, \; j = 1,3$, that might be collected into a 3×3 matrix as follows

$$\mathbf{E} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}. \tag{IN\_21}$$

Due to the tensor symmetry, i.e. $E_{12} = E_{21}, E_{13} = E_{31}, E_{23} = E_{32}$, the Voigt’s notation is often used. The geometrical and physical meanings of the strain tensor components (sometimes simply called strains) are derived, explained and presented in the Paragraph 03_ST. Stress.

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{Bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{12} \\ 2E_{23} \\ 2E_{31} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}. \qquad \text{(IN\_22)}$$

Notice that in the case of strains, there is no one-to-one correspondence between mathematical and engineering components, as it was shown before for the stress components. The 'strange' appearance of factor 2 is due energy considerations and will be explained later.

1.3.7. Principal axes and invariants of the second order tensor

The components T_{ij} of the second order tensor $\mathbf{T}$ in the coordinate system $\mathbf{x} = x_1, x_2, x_3$

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \qquad \text{(IN\_23)}$$

could be expressed in another, say primed, coordinate system $\mathbf{x}' = x'_1, x'_2, x'_3$ whose position with respect to the original one is given by the rotation around the common origin as depicted in Fig. IN_9.

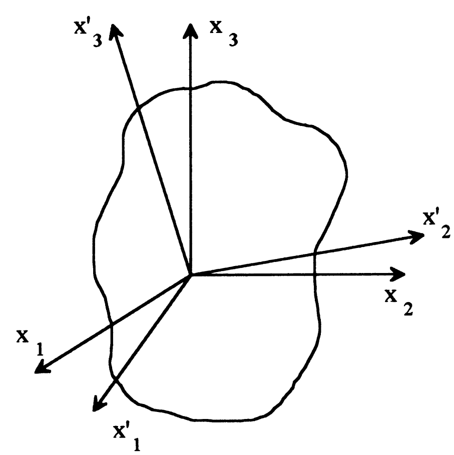


Fig. IN_9 … Rotated axes

Such a rotation is described by the rotation matrix, say $\mathbf{A}$, of direction cosines $a_{ij} = \cos(\angle x'_i x_j)$. The argument $\angle x'_i x_j$ represents the angle between the $x'_i - th$ axis of the rotated (i.e. primed) system with respect to $x_j - th$ axis of the original system.

Obviously, the components of the tensor in the rotated system

$$\mathbf{T}' = \begin{bmatrix} T'_{11} & T'_{12} & T'_{13} \\ T'_{21} & T'_{22} & T'_{23} \\ T'_{31} & T'_{32} & T'_{33} \end{bmatrix} \qquad \text{(IN\_24)}$$

are obtained by the transformation relation in the form

$$\mathbf{T}' = \mathbf{A}\mathbf{T}\mathbf{A}^{\mathrm{T}} . \qquad \text{(IN\_25)}$$

Now, we are looking for such a rotation – that as the result of the transformation Eq. (IN_25) – produces the diagonal form of the tensor. Meaning, that the all of the out-of diagonal components vanish. And, the orientation of a new coordinate system – in which the original tensor becomes diagonal – is determined by angles $\angle x_i' x_j \quad i = 1:3, j = 1:3$.

Mathematically, this task leads to a so-called standard eigenvalue problem which is defined by

$$(\mathbf{T} - \lambda \mathbf{I})\mathbf{a} = \mathbf{0} . \qquad \text{(IN\_26)}$$

The scalar λ contains the eigenvalues, while the vector **a** contains the eigenvectors. The eigenvalues of the tensor are also called the principal values. The eigenvector contains the corresponding direction cosines of the angles $\angle x_i' x_j \quad i = 1:3, j = 1:3$. It appears that the solution is not unique – there are as many eigenvalues as is the rank of the tensor, also there are as many eigenvectors. In this case, for the tensors of the second order, we have three eigenvalues and three eigenvectors. If the tensor is symmetric[8], all the eigenvalues are real and all the eigenvectors are orthogonal.

The Eq. (IN_26) represents the system of homogeneous equations, which has a unique solution only if the determinant of the system matrix is equal to zero, i.e.

$$\det(\mathbf{T} - \lambda \mathbf{I}) = 0 . \qquad \text{(IN\_27)}$$

Evaluating the determinant we get a cubic equation

$$\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0 , \qquad \text{(IN\_28)}$$

whose roots $\lambda_i \quad i = 1:3$. For symmetric tensors, we are mainly dealing with, the roots are always real. Historically, the Cardan's formula was used for the solution of cubic equations. In Matlab, the built-in function `root` might be used for this task. To find the eigenvalues and eigenvectors of the matrix representing the tensor, the built-in function `eig` is used.

The coefficients appearing by the individual powers of λ are called the tensor invariants

$$I_1 = T_{11} + T_{22} + T_{33} , \qquad \text{(IN\_29)}$$

[8] And the stress and strain tensors have this property.

$$I_2 = \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{vmatrix}, \tag{IN\_30}$$

$$I_3 = \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix}. \tag{IN\_31}$$

Do you remember how a determinant is evaluated by hand?

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb, \tag{IN\_32}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - hf) + b(fg - di) + c(dh - eg). \tag{IN\_33}$$

To summarize. In continuum mechanics, we deal with the second order strain and stress tensors.

- The eigenvalues of the strain tensor $\mathbf{E} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}$ are called the *principal strains* and might be denoted E_1, E_2, E_3.

- The strain invariants are
    - $I_1 = E_{11} + E_{22} + E_{33}$, (IN_34)
    - $I_2 = \begin{vmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{vmatrix} + \begin{vmatrix} E_{11} & E_{13} \\ E_{31} & E_{33} \end{vmatrix} + \begin{vmatrix} E_{22} & E_{23} \\ E_{32} & E_{33} \end{vmatrix}$, (IN_35)
    - $I_3 = \begin{vmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{vmatrix}$. (IN_36)

- The eigenvalues of the stress tensor $\mathbf{\Sigma} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{bmatrix}$ are called the *principal stresses* and might be denoted $\Sigma_1, \Sigma_2, \Sigma_3$.

- The stress invariants are
    - $J_1 = \Sigma_{11} + \Sigma_{22} + \Sigma_{33}$, (IN_37)

- $J_2 = \begin{vmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{vmatrix} + \begin{vmatrix} \Sigma_{11} & \Sigma_{13} \\ \Sigma_{31} & \Sigma_{33} \end{vmatrix} + \begin{vmatrix} \Sigma_{22} & \Sigma_{23} \\ \Sigma_{32} & \Sigma_{33} \end{vmatrix},$ (IN_38)
- $J_3 = \begin{vmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{vmatrix}.$ (IN_39)

Remark: Sometimes it is convenient to decompose the stress tensor into the volumetric and deviatoric parts as follows

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{bmatrix} = \underbrace{\begin{bmatrix} \Sigma_{\mathrm{m}} & & \\ & \Sigma_{\mathrm{m}} & \\ & & \Sigma_{\mathrm{m}} \end{bmatrix}}_{\text{volumetric}} + \underbrace{\begin{bmatrix} \Sigma_{11}-\Sigma_{\mathrm{m}} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22}-\Sigma_{\mathrm{m}} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33}-\Sigma_{\mathrm{m}} \end{bmatrix}}_{\text{deviatoric}}, \quad \text{(IN\_40)}$$

where the mean stress is $\Sigma_{\mathrm{m}} = \frac{1}{3}\left(\Sigma_{11} + \Sigma_{22} + \Sigma_{33}\right).$ (IN_41)

The volumetric part of stress is responsible for changes of volume only. The deviatoric part of stress causes the change of shape only.

Example – principal stresses and strains

Given: The Young modulus and Poisson ratio are $E = 2.1 \times 10^{11}$ Pa, $\mu = 0.3$. The stress

components are $\mathbf{\Sigma} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{bmatrix} = \begin{bmatrix} 40 & 20 & -10 \\ 20 & -35 & -15 \\ -10 & -15 & 60 \end{bmatrix}.$ (IN_42)

Determine: Principal stresses and strains in Matlab.

The stress matrix is

```
sig = [40  20 -10;
       20 -35 -15;
      -10 -15  60];
```

The eigenvalues `Lambda` and the eigenvectors `V` are obtained by the statement

```
[V,Lambda] = eig(sig);
```

The eigenvalues obtained by the Matlab procedure are sorted by magnitude. To sort them in descending order, to get the principal stresses $\Sigma_1, \Sigma_2, \Sigma_3$, we might write

```
Lsort = sort(Lambda,'descend');       % sort in descending order
SIG = Lsort';                 % principal stresses Σ1,Σ2,Σ3
```

The principal strains could be obtained as follows

$$\mathbf{E} = \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix} = \underbrace{\frac{1}{E}\begin{bmatrix} 1 & -\mu & -\mu \\ -\mu & 1 & -\mu \\ -\mu & -\mu & 1 \end{bmatrix}}_{\mathbf{D}} \begin{Bmatrix} \Sigma_1 \\ \Sigma_2 \\ \Sigma_3 \end{Bmatrix}. \qquad \text{(IN\_43)}$$

Define D matrix in Matlab

```
Dmatrix = [1 -mi  -mi;
          -mi   1  -mi;
          -mi -mi    1]/E;
```

Evaluating the product `EPS = Dmatrix*SIG` you will get the principal strains

```
EPS =
  1.0e-009 *

    0.3332 0.1403 -0.3496
```

In the mathematical theory of elasticity, the strain energy is defined as the double dot product of the stress and strain tensors, i.e. $s = \frac{1}{2}\Sigma_{ij}\ E_{ij}$. Of course, the physical quantity, i.e. the strain energy, should not depend on the notation being used. So, to get the same result and to take into account that in engineering style the symmetric components of strain are only taken once, we have to express it as a dot product of arrays in the form $s = \frac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\varepsilon}$.

1.3.8. Examples

Example – Kronecker delta, a unit matrix

The orthogonality of the matrix of direction cosines can be expressed by means of so-called the Kronecker delta, alternatively by means of the unit matrix[9] as follows

$$R_{ij}R_{ik} = \delta_{jk} \qquad \text{or} \qquad \mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}, \qquad \text{(IN\_44)}$$

where

$$\delta_{jk} = \begin{cases} 1 & \text{for} \quad j = k \\ 0 & \text{for} \quad j \neq k \end{cases} \qquad \text{and} \qquad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad \text{(IN\_45)}$$

[9] Also called the identity matrix.

The Kronecker delta is often used as the *substitution operator* allowing to express

$$\delta_{ik} x_k = x_i \quad or \quad \mathbf{Ix} = \mathbf{x}\,, \tag{IN\_46}$$

since it has the effect of renaming indices.

Example – tensor of the third order – Levi-Civita permutation operator, $N = 3$

It is convenient to introduce the Levi-Civita tensor which is defined by

$$\in_{ijk} = \begin{cases} 1 & \text{for even permutation of indices: } 1,2,3 \;\; 2,3,1 \;\; 3,1,2 \\ 0 & \text{for repeated indices as: } 1,1,2 \text{ etc} \\ -1 & \text{for odd permutation of indices: } 3,2,1 \;\; 2,1,3 \;\; 1,3,2 \end{cases} . \tag{IN\_47}$$

This tensor serves mainly for expressing the cross product operation (that actually does not belong to the menagerie of the tensor calculus, but is frequently used in mechanics) as follows

$$\vec{c} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \vec{a} \times \vec{b} = \begin{Bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{Bmatrix} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \qquad \text{or} \qquad c_i = \in_{ijk} a_j b_k \,.$$

… (IN_48)

1.3.9. Implementations of basic matrix and tensor operations

Tensor addition and subtraction

This operation is defined for tensors of the same order only. It is provided element by element as

$$T_{ij} = A_{ij} \pm B_{ij}\,.$$

Tensor contraction

is a process in which two initially differently named dummy indices, say i, j, are replaced by one of the previously used letters – say i or j. By the contraction operation, the order of the tensor order is decreased by two.

Example – tensor contractions

$$T_{ij} \to T_{ii} \to s,$$
$$R_{ijk} \to R_{ijj} \to v_i,$$
$$U_{ijkl} \to U_{ijjl} \to S_{il}.$$

The programming equivalent of the second presented item is shown in Matlab program 4.

```
% Matlab program 4
% tensor contraction for n = 3

for i = 1:n
  v(i) = 0;
  for j = 1:n
    v(i) = v(i) + R(i,j,j);   % compare with index notation, i.e. R_ijk -> R_ijj -> v_i
  end
end
```

Three cases of multiplication of tensors of the first order – vectors

1. *Dot product of vectors – sometimes called the scalar product*

- Index notation $s = a_i b_i$.
- Symbolic notation $s = \vec{a} \cdot \vec{b}$.
- Matrix notation (for column vectors) $s = \mathbf{a}^{\mathrm{T}}\mathbf{b} = \{a_1\ a_2\ a_3\}\begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$.
- Matlab command `s = dot(a,b).`

In Matlab, we could proceed as shown in Matlab program 5.

```
% Matlab program 5
% vector dot product for n = 3

sum = 0;
  for i = 1:n
    sum = sum + a(i) * b(i); % % compare with index notation, i.e. s = a_i b_i
  end;
s = sum;
```

2. *Dyadic product of vectors*

- Index notation $C_{ij} = a_i b_j$.
- Symbolic notation $\mathbf{C} = \vec{a} \otimes \vec{b} = \mathbf{a} \otimes \mathbf{b}$.
- Matrix notation (for column vectors[10])

[10] In this, and in the following examples, it is assumed that the vectors are of column nature.

$$\mathbf{C} = \mathbf{a}\mathbf{b}^{\mathrm{T}} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \{ b_1 \ b_2 \ b_3 \} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}.$$

- Matlab command (for column vectors) `C = a*b'.`

In Matlab, we could proceed as follows

```
% Matlab program 6
% vector dyadic product for n = 3

for i = 1:n
  for j = 1:n
    C(i,j) = a(i) * b(j); % compare with index notation, i.e. C_ij = a_i b_j
  end
end
```

3. *Cross product, sometimes called the vector product, valid for* $n = 3$ *only*

- Index notation $c_i = \in_{ijk} a_j b_k$
- Symbolic notation $\vec{c} = \vec{a} \times \vec{b} = \mathbf{a} \times \mathbf{b}$
  - Matrix notation (Sarus rule evaluation)

$$\mathbf{c} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{Bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{Bmatrix}$$

- Matlab command `c = cross(a,b)`

Three cases of multiplication of tensors of the second order

1. *Tensor double dot product*

- Index notation $s = A_{ij} B_{ij}$
- Symbolic notation $s = \mathbf{A} : \mathbf{B}$
- Matrix notation $s = \mathrm{tr}(\mathbf{A}^{\mathrm{T}} \mathbf{B})$
- Matlab command `s = trace(A'*B)`

The matrix operator **tr** signifies the trace of a matrix. The algorithm is in the Matlab program 7.

```
% Matlab program 7
% double dot product of the second-order tensors for n = 3

s = 0;
for i = 1:n
  for j = 1:n
    s = s + A(i,j)*B(i,j); % … compare with index notation, i.e. s = A_ij B_ij
  end
end
```

2. *Tensor and matrix multiplication*

- Index notation $C_{ij} = A_{ik} B_{kj}$
- Symbolic notation $\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$
- Matrix notation $\mathbf{C} = \mathbf{AB}$
- Matlab command `C = A*B`

For tensors of the second order and for 3×3 matrices we could express this operation as indicated in Matlab program 8.

```
% Matlab program 8
% matrix multiplication for n = 3

for i = 1:n
  for j = 1:n
    C(i,j) = 0;
    for k = 1:n
      C(i,j) = C(i,j) + A(i,k)*B(k,j); % … compare with index notation, i.e. C_ij = A_ik B_kj
    end
  end
end
```

3. *Dyadic product of second-order tensors*

- Index notation $C_{ijkl} = A_{ij} B_{kl}$
- Symbolic notation $\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$
- Matrix notation --
- Matlab command --

The algorithm is in the Matlab program 9.

```
% Matlab program 9
% dyadic products of second-order tensors for n = 3

for i = 1:n
  for j = 1:n
    for k = 1:n
      for l = 1:n
        C(i,j,k,l) = A(i,j)*B(k,l); % … compare with index notation, i.e. C_ijkl = A_ij B_kl
      end
    end
  end
end
```

Notice, that index notation represents a direct hint for the evaluation of above formulas, while the symbolic and matrix representations are really symbolic – we have to remember and understand the assumed meanings of operations, operators and accompanying symbols.

For more details see [14], [15], [24], [25], [32], [36].

02_KI. Kinematics

2.1. Deformation and strain

Kinematics studies the motion of bodies without being interested in the causes inducing that motion. We will limit our attention to the analysis of individual material points (particles) of solid bodies that are being deformed. The bodies thus change their positions in space, their volumes and, consequently, individual material particles change their positions. This process is called *deformation*. The motion of an individual particle is quantified by two distinct ways. First, by measuring the change of the position of each material particle with respect to a fixed coordinate system – this quantity is called *displacement* and is measured in [m]. Second, by that displacement normalized with respect to a suitably chosen reference distance. This quantity is called *strain* and is dimensionless.

The analysis of the particle motion requires distinguishing two types of coordinates. Namely, the *material coordinates* labeling the material particle, and the *spatial coordinates* indicating the current position of that particle.

2.2. Material and spatial coordinates – configuration

Consider a solid body occupying at a given time a finite spatial region. Assume that the region is completely filled up by a continuously distributed matter. See Fig. KI_1.

The position of each infinitesimally small material particle, say P, is uniquely determined by the instantaneous spatial coordinates of that particle.

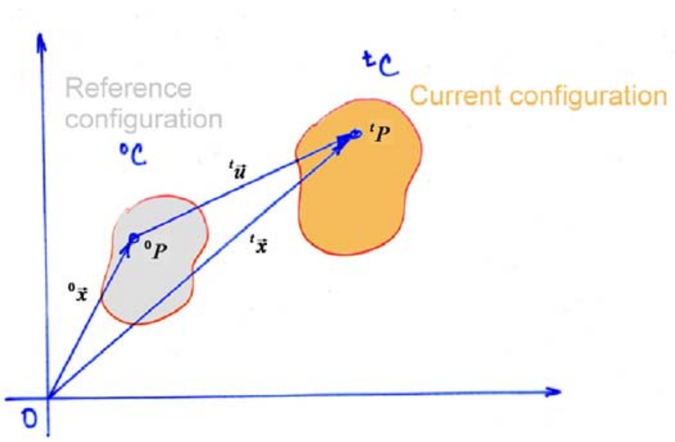


Fig. KI_1 ... Kinematics configuration

The initial configuration of the body at the time $t = 0$ is denoted ${}^0C$ and is called the *initial* or *reference* configuration.

Later, at a generic time t, the body is moved and deformed at the same time. After the deformation process, the body occupies a new configuration, say ${}^tC$. The coordinates of the considered particle P in the configuration ${}^0C$ are ${}^0x_i$ and might be denoted by an identifier ${}^0P$. The position of the same particle, i.e. P, in the deformed or current configuration ${}^tC$, i.e. at the time t, is defined by coordinates ${}^tx_i$ and might be denoted by an identifier ${}^tP$.

Often, the term *material point* is used as the synonym for the *material particle*.

Notation and terminology used in Fig. KI_1 is as follows

${}^{0}x_i$	material or Lagrangian coordinate, position of material particle at time $t = 0$,
${}^{t}x_i$	spatial or Eulerian coordinate, position of material particle at time t,
${}^{t}u_i = {}^{t}x_i - {}^{0}x_i$	displacement of material particle at time t.

The vector ${}^{t}\vec{u} = {}^{t}\vec{x} - {}^{0}\vec{x}$ – or written alternatively as ${}^{t}u_i = {}^{t}x_i - {}^{0}x_i$ or ${}^{t}\mathbf{u} = {}^{t}\mathbf{x} - {}^{0}\mathbf{x}$ – is the measure of the difference of positions of the material particle P before and after the deformation process. This vector is called the *displacement*.

There is an alternative notation used in literature:

${}^{0}x_i, X_i$	as components of ${}^{0}\mathbf{x}, \mathbf{X}$	or	$\vec{x}, \vec{X}$	for Lagrangian coordinates,
${}^{t}x_i, x_i$	as components of ${}^{t}\mathbf{x}, \mathbf{x}$	or	${}^{t}\vec{x}, \vec{x}$	for Eulerian coordinates.

See [14], [18], [19], [23], [32], [36].

2.3. Lagrangian and Eulerian formulations of deformation

The function prescribing the motion of a material particle between the reference (initial), i.e. ${}^{0}C$, and the current, i.e. ${}^{t}C$, configurations can be expressed by a function

$$ {}^{t}x_i = f_i\left({}^{0}x_j, t\right) = {}^{t}x_i\left({}^{0}x_j, t\right). \qquad \text{(KI\_1)} $$

For brevity, instead of a generic functional operator f, we are using the variable name[1].

This relation, called the *Lagrangian formulation of deformation* or the *Lagrangian transformation*, prescribes the positions of a particular material particle, as a function of its initial position and time. Generally, this function is different for each material particle. This relation prescribes the history of individual material particles in time and space.

The inverse function to that prescribed by Eq. (KI_1) is

$$ {}^{0}x_i = {}^{0}x_i\left({}^{t}x_j, t\right) \qquad \text{(KI\_2)} $$

and is called the *Eulerian formulation of deformation* or the *Eulerian transformation*. It prescribes the sequence of displacements of different material points (particles) as they pass through the particular point in space.

While the Lagrangian formulation is currently used in solid mechanics, the Eulerian formulation, prescribed by Eq. (KI_2), is preferred in fluid mechanics.

[1] Recall that we often write $y = y(x)$ instead of $y = f(x)$.

Both formulations, being applied to the same physical phenomenon, should give identical results. This condition is satisfied if and only if the functions Eq. (KI_1) and Eq. (KI_2) are mutually invertible.

Mathematically, this condition requires that the *Jacobian of the transformation*, given by Eq. (KI_1), is non-zero, thus

$$J = \det({}_0^tF_{ij}) \neq 0\,, \qquad \text{(KI\_3)}$$

where

$${}_0^tF_{ij} = \frac{\partial\, {}^tx_i}{\partial\, {}^0x_j}\,. \qquad \text{(KI\_4)}$$

The quantity ${}_0^tF_{ij}$ is called the *deformation gradient*, sometimes *material deformation gradient*. In 3D it is composed of nine elements. Generally, it is unsymmetrical. The upper left and the lower left indices indicate that the deformation gradient is defined in the current configuration ${}^tC$ and is related to the reference configuration ${}^0C$. Later, we will prove that the Jacobian for a physically attainable deformation is not only non-zero but is furthermore positive and finite, i.e. $0 < J < \infty$. This condition physically, or rather geometrically, means that the volume of the body being deformed will not become zero or infinite, that there are no gaps within the considered volume. Furthermore, this condition guarantees that two initially distinct material particles will not end up in a single spatial point – this way it is secured that no material penetration can occur.

2.4. Deformation and strain

The term *deformation* semantically means the change of shape. In engineering, the deformation of a solid body is analyzed by measuring the *displacements* of material particles. This measure, considered in meters, says nothing about the magnitude of displacements – weather they are infinitesimal, small, or finite. Defining small displacements is crucial since the linear theory of elasticity is based on it. That's why there are defined additional measures of deformation. They are called *strains* and are obtained by normalizing the analyzed displacements with respect to suitable distances, somehow related to the size of the body. Thus, the *strain* is a dimensionless quantity that is independent of the size of the examined body.

The proper definition of strain quantities requires that they are independent of the orientation of the coordinate system and independent of the rigid body motions. We will show that not all the strain measures used in engineering are endowed by this quality.

There are infinitely many ways how to define a 'good' strain measure that satisfies the condition of its independence on the coordinate system and at the same its invariance to the rigid body motion. One of them, being invented by our forefathers, is based on the fact that it is the length of a line segment which is independent of the choice of the coordinate system.

To derive a suitable strain measure, observe positions of two material line segments depicted in Fig. KI_2. For simplicity, the situation is depicted in 2D space, but the corresponding geometrical and mathematical reasoning is considered in 3D. The elementary line segment connects two material particles, say P and Q. In the reference configuration ${}^0C$ they are located at spatial points denoted by ${}^0P$ and ${}^0Q$. This material line segment is represented by the vector $\mathrm{d}^0\mathbf{x}$ whose length is $\left|\mathrm{d}^0\mathbf{x}\right| = \mathrm{d}^0 s$. Due to the deformation, this material line segment is elongated and moved into a new position, which is denoted as the configuration ${}^tC$. The material particles P and Q are now located at spatial points ${}^tP$ and ${}^tQ$ and the vector representing the material line segment is $\mathrm{d}^t\mathbf{x}$ and its length is $\left|\mathrm{d}^t\mathbf{x}\right| = \mathrm{d}^t s$.

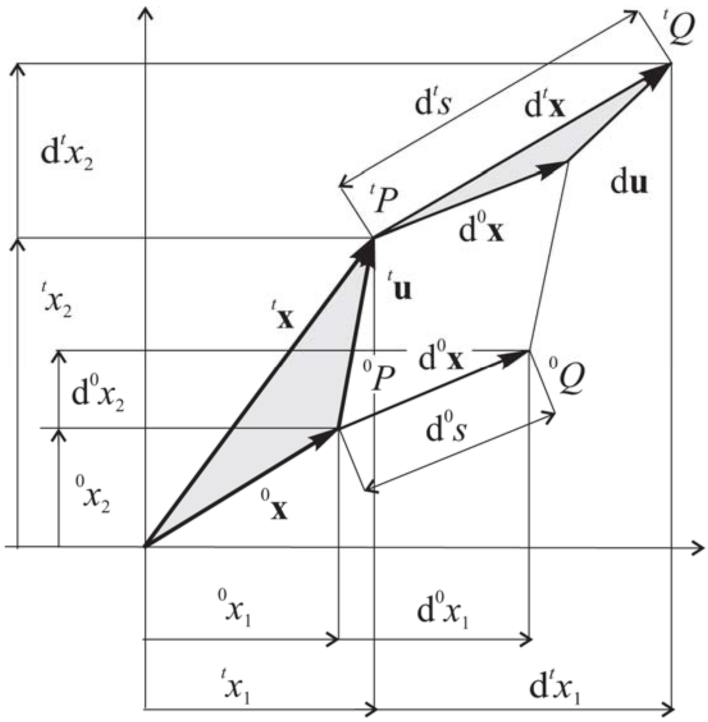


Fig. KI_2 ... Material line segments

The displacement of the material point P is

$${}^t u_i = {}^t x_i - {}^0 x_i .$$

Differentiating the previous relation with respect to the material coordinate we get the relation suitable for the future reasoning, i.e.

$$\frac{\mathrm{d}^t u_i}{\mathrm{d}^0 x_j} = \frac{\mathrm{d}^t x_i}{\mathrm{d}^0 x_j} - \delta_{ij} \qquad \text{or} \qquad {}^t_0 Z_{ij} = {}^t_0 F_{ij} - \delta_{ij} \qquad \text{or} \qquad {}^t_0\mathbf{Z} = {}^t_0\mathbf{F} - \mathbf{I}, \qquad \text{(KI\_5)}$$

where the term $\dfrac{\mathrm{d}^t u_i}{\mathrm{d}^0 x_j}$, denoted as ${}^t_0 Z_{ij}$ or ${}^t_0\mathbf{Z}$, is called the *material displacement gradient*. The deformation gradient was already defined as ${}^t_0 F_{ij} = \dfrac{\mathrm{d}^t x_i}{\mathrm{d}^0 x_j}$.

To summarize. The coordinates of material particles P and Q in the configuration ${}^0C$ are

$$\begin{aligned} {}^0P&: \quad {}^0x_i, \\ {}^0Q&: \quad {}^0x_i + \mathrm{d}\,{}^0x_i. \end{aligned}$$

The coordinates of the same material particles P and Q in the configuration ${}^tC$ are

$$\begin{aligned} {}^tP&: \quad {}^tx_i = {}^0x_i + {}^tu_i, \\ {}^0Q&: \quad {}^tx_i + \mathrm{d}\,{}^tx_i = {}^0x_i + {}^tu_i + \frac{\mathrm{d}\,{}^tx_i}{\mathrm{d}\,{}^0x_j}\mathrm{d}\,{}^0x_j = {}^0x_i + {}^tu_i + {}_0^tF_{ij}\,\mathrm{d}\,{}^0x_j. \end{aligned}$$

We have used the fact that the deformed line segment can be expressed as the first order differential $\mathrm{d}\,{}^tx_i = \frac{\mathrm{d}\,{}^tx_i}{\mathrm{d}\,{}^0x_j}\mathrm{d}\,{}^0x_j$ and that ${}_0^tF_{ij} = \frac{\mathrm{d}\,{}^tx_i}{\mathrm{d}\,{}^0x_j}$. This way we came to an important conclusion, namely that knowing the transformation ${}^0x_i = {}^0x_i\left({}^tx_j, t\right)$ and being able to evaluate the deformation gradient ${}_0^tF_{ij}$, that satisfies the condition $\det {}_0^tF_{ij} \neq 0$, we have at our disposal the formula determining the deformed segment line, i.e. $\mathrm{d}\,{}^tx_i = {}_0^tF_{ij}\,\mathrm{d}\,{}^0x_j$. In other words: the deformation gradient is an operator that being applied to the material segment line in the reference configuration ${}^0C$ gives its description in the current configuration ${}^tC$.

Thus, the formula describing the deformation of the elementary material line segment from the configuration ${}^0C$ to ${}^tC$ is

$$\mathrm{d}\,{}^tx_i = {}_0^tF_{ij}\,\mathrm{d}\,{}^0x_j \qquad \text{or} \qquad \mathrm{d}\,{}^t\mathbf{x} = {}_0^t\mathbf{F}\,\mathrm{d}\,{}^0\mathbf{x}. \tag{KI\_6}$$

Similarly, using the *material displacement gradient*, we define the *material displacement increment*. It is depicted in Fig. KI_2.

$$\mathrm{d}u_i = \frac{\partial\,{}^tu_i}{\partial\,{}^0x_j}\mathrm{d}\,{}^0x_j = {}_0^tZ_{ij}\,\mathrm{d}\,{}^0x_j \qquad \text{or} \qquad \mathrm{d}\mathbf{u} = {}_0^t\mathbf{Z}\,\mathrm{d}\,{}^0\mathbf{x}. \tag{KI\_7}$$

The lengths of the considered material line segments before and after the deformation are expressed by means of the Pythagorean Theorem.

$$\begin{aligned} {}^0C&: \quad \mathrm{d}\,{}^0s = \left(\mathrm{d}\,{}^0x_i\,\mathrm{d}\,{}^0x_i\right)^{\frac{1}{2}}, \\ {}^tC&: \quad \mathrm{d}\,{}^ts = \left(\mathrm{d}\,{}^tx_i\,\mathrm{d}\,{}^tx_i\right)^{\frac{1}{2}}. \end{aligned}$$

The length of a segment line is invariant with respect to a choice of the coordinate system; the same applies to squares of lengths and to their differences as well. So, the difference of squares of lengths of the same material line segment, before and after the deformation is

$$\left(\mathrm{d}\,{}^ts\right)^2 - \left(\mathrm{d}\,{}^0s\right)^2 = \mathrm{d}\,{}^tx_i\,\mathrm{d}\,{}^tx_i - \mathrm{d}\,{}^0x_i\,\mathrm{d}\,{}^0x_i = \mathrm{d}\,{}^t\mathbf{x}^{\mathrm{T}}\,\mathrm{d}\,{}^t\mathbf{x} - \mathrm{d}\,{}^0\mathbf{x}^{\mathrm{T}}\,\mathrm{d}\,{}^0\mathbf{x}. \tag{KI\_8}$$

Working with squares is advantageous since it relieves us of dealing with square roots.

2.5. Green-Lagrange strain tensor

Eq. (KI_1) represents a strain measure that is invariant with respect to the choice of the coordinate system. This could be further elaborated. Expressing that difference means of the reference coordinates we get

$$\left(\mathrm{d}^t s\right)^2-\left(\mathrm{d}^0 s\right)^2=\mathrm{d}^0\mathbf{x}^{\mathrm{T}}\ {}_0^t\mathbf{F}^{\mathrm{T}}\ {}_0^t\mathbf{F}\ \mathrm{d}^0\mathbf{x}-\mathrm{d}^0\mathbf{x}^{\mathrm{T}}\ \mathbf{I}\ \mathrm{d}^0\mathbf{x}=2\,\mathrm{d}^0\mathbf{x}^{\mathrm{T}}\ \underbrace{\frac{1}{2}\left({}_0^t\mathbf{F}^{\mathrm{T}}\ {}_0^t\mathbf{F}-\mathbf{I}\right)}_{{}_0^t\mathbf{E}^{\mathrm{GL}}}\mathrm{d}^0\mathbf{x}. \qquad \text{(KI\_9)}$$

We have obtained the difference of squares of lengths of the material line segment as a scalar quantity having the form of the *quadratic form of variables*. The trick with $\frac{1}{2}$ and 2 factors will be explained later. The middle part of this difference, i.e. ${}_0^t\mathbf{E}^{\mathrm{GL}}$, is a quantity called the *Green-Lagrange strain tensor*[2]. It is obvious that the difference of lengths could only be zero (if the rigid body motion) or positive. The tensor for which the quadratic form of variables – for any nonzero vectors – is positive is said to be *positive definite*.

The Green-Lagrange strain tensor is related to configuration ${}^tC$ and expressed by coordinates of the configuration ${}^0C$. It can be presented in various forms.

$$ {}_0^t\mathbf{E}^{\mathrm{GL}}=\frac{1}{2}\left({}_0^t\mathbf{Z}+{}_0^t\mathbf{Z}^{\mathrm{T}}+{}_0^t\mathbf{Z}^{\mathrm{T}}\,{}_0^t\mathbf{Z}\right),$$

$$ {}_0^tE_{ij}^{\mathrm{GL}}=\frac{1}{2}\left({}_0^tZ_{ij}+{}_0^tZ_{ji}+{}_0^tZ_{ki}\ {}_0^tZ_{kj}\right)=\frac{1}{2}\left(\frac{\partial^t u_i}{\partial^0 x_j}+\frac{\partial^t u_j}{\partial^0 x_i}+\frac{\partial^t u_k}{\partial^0 x_i}\frac{\partial^t u_k}{\partial^0 x_j}\right). \qquad \ldots \text{(KI\_10)}$$

One could notice that the Green-Lagrange strain tensor is composed of two parts. The former contains the derivatives of the first order, while in the latter there are derivatives of the second order and the products of derivatives of the first order.

If the higher order terms could be neglected then the Green-Lagrange strain tensor becomes the infinitesimal Cauchy strain tensor. More about it later.

[2] It was derived in 1841 by George Green and independently by Saint Venant in 1844. Sometimes, it is called simply the Green strain tensor.

The Green-Lagrange strain tensor is symmetric and has nine components in 3D space. Expressing them in full we get

$$
\begin{aligned}
{}_0^tE^{\mathrm{GL}}_{11} &= \frac{\partial^t u_1}{\partial^0 x_1} + \frac{1}{2}\left[\left(\frac{\partial^t u_1}{\partial^0 x_1}\right)^2 + \left(\frac{\partial^t u_2}{\partial^0 x_1}\right)^2 + \left(\frac{\partial^t u_3}{\partial^0 x_1}\right)^2\right],\\
{}_0^tE^{\mathrm{GL}}_{22} &= \frac{\partial^t u_2}{\partial^0 x_2} + \frac{1}{2}\left[\left(\frac{\partial^t u_1}{\partial^0 x_2}\right)^2 + \left(\frac{\partial^t u_2}{\partial^0 x_2}\right)^2 + \left(\frac{\partial^t u_3}{\partial^0 x_2}\right)^2\right],\\
{}_0^tE^{\mathrm{GL}}_{33} &= \frac{\partial^t u_3}{\partial^0 x_3} + \frac{1}{2}\left[\left(\frac{\partial^t u_1}{\partial^0 x_3}\right)^2 + \left(\frac{\partial^t u_2}{\partial^0 x_3}\right)^2 + \left(\frac{\partial^t u_3}{\partial^0 x_3}\right)^2\right],\\
{}_0^tE^{\mathrm{GL}}_{12} = {}_0^tE^{\mathrm{GL}}_{21} &= \frac{1}{2}\left[\frac{\partial^t u_1}{\partial^0 x_2} + \frac{\partial^t u_2}{\partial^0 x_1}\right] + \frac{1}{2}\left[\frac{\partial^t u_1}{\partial^0 x_1}\frac{\partial^t u_1}{\partial^0 x_2} + \frac{\partial^t u_2}{\partial^0 x_1}\frac{\partial^t u_2}{\partial^0 x_2} + \frac{\partial^t u_3}{\partial^0 x_1}\frac{\partial^t u_3}{\partial^0 x_2}\right],\\
{}_0^tE^{\mathrm{GL}}_{23} = {}_0^tE^{\mathrm{GL}}_{32} &= \frac{1}{2}\left[\frac{\partial^t u_2}{\partial^0 x_3} + \frac{\partial^t u_3}{\partial^0 x_2}\right] + \frac{1}{2}\left[\frac{\partial^t u_1}{\partial^0 x_2}\frac{\partial^t u_1}{\partial^0 x_3} + \frac{\partial^t u_2}{\partial^0 x_2}\frac{\partial^t u_2}{\partial^0 x_3} + \frac{\partial^t u_3}{\partial^0 x_2}\frac{\partial^t u_3}{\partial^0 x_3}\right],\\
{}_0^tE^{\mathrm{GL}}_{31} = {}_0^tE^{\mathrm{GL}}_{13} &= \frac{1}{2}\left[\frac{\partial^t u_3}{\partial^0 x_1} + \frac{\partial^t u_1}{\partial^0 x_3}\right] + \frac{1}{2}\left[\frac{\partial^t u_1}{\partial^0 x_3}\frac{\partial^t u_1}{\partial^0 x_1} + \frac{\partial^t u_2}{\partial^0 x_3}\frac{\partial^t u_2}{\partial^0 x_1} + \frac{\partial^t u_3}{\partial^0 x_3}\frac{\partial^t u_3}{\partial^0 x_1}\right].
\end{aligned}
\qquad \text{… (KI\_11)}
$$

The Green-Lagrange strain components in the Voigt's notation are

$$
\boldsymbol{\varepsilon}^{\mathrm{GL}} = \begin{Bmatrix} \varepsilon_1^{\mathrm{GL}} \\ \varepsilon_2^{\mathrm{GL}} \\ \varepsilon_3^{\mathrm{GL}} \\ \varepsilon_4^{\mathrm{GL}} \\ \varepsilon_5^{\mathrm{GL}} \\ \varepsilon_6^{\mathrm{GL}} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^{\mathrm{GL}} \\ \varepsilon_{yy}^{\mathrm{GL}} \\ \varepsilon_{zz}^{\mathrm{GL}} \\ \gamma_{xy}^{\mathrm{GL}} \\ \gamma_{yz}^{\mathrm{GL}} \\ \gamma_{zx}^{\mathrm{GL}} \end{Bmatrix} = \begin{Bmatrix} E_{11}^{\mathrm{GL}} \\ E_{22}^{\mathrm{GL}} \\ E_{33}^{\mathrm{GL}} \\ 2E_{12}^{\mathrm{GL}} \\ 2E_{23}^{\mathrm{GL}} \\ 2E_{31}^{\mathrm{GL}} \end{Bmatrix}. \qquad \text{… (KI\_12)}
$$

2.6. Almansi strain tensor

The Green-Lagrange strain tensor was derived by excluding the ${}^t x_i$ coordinate from the expression given by Eq. (KI_8) describing the difference of squares of lengths of a material line segment. A different strain measure, called the Almansi strain tensor, can be obtained by excluding the coordinate ${}^0 x_i$ from Eq. (KI_8) instead.

Using the inverse relation to that expressed by Eq. (KI_6) that describes the transformation ${}^tC \to {}^0C$, namely

$$
\mathrm{d}^0\mathbf{x} = {}_0^t\mathbf{F}^{-1}\mathrm{d}^t\mathbf{x}. \qquad \text{(KI\_13)}
$$

The quantity ${}_0^t\mathbf{F}^{-1}$ represents the *inverse of the material deformation gradient*. Formally we write

$$ {}_0^t\mathbf{F}^{-1} = {}_t^0\mathbf{F} = \frac{\partial\, {}^0x_i}{\partial\, {}^tx_j}\,. \tag{KI\_14} $$

The inverse of the *material deformation gradient* ${}_0^t\mathbf{F}^{-1}$ is denoted ${}_t^0\mathbf{F}$ and called the *spatial deformation gradient*. Of course, the condition for its existence is the non-zero value of the Jacobian of the transformation, i.e.

$$ J = \det {}_0^t\mathbf{F} \neq 0\,. \tag{KI\_15} $$

As before, we differentiate the relation for displacements ${}^tu_i = {}^tx_i - {}^0x_i$, this time with respect to spatial coordinates

$$ \frac{\partial\, {}^tu_i}{\partial\, {}^tx_j} = \delta_{ij} - \frac{\partial\, {}^0x_i}{\partial\, {}^tx_j} \qquad \text{or} \qquad {}_t^t\overline{\mathbf{Z}} = \mathbf{I} - {}_0^t\mathbf{F}^{-1}\,. \tag{KI\_16} $$

We have defined a new variable, i.e.

$$ {}_t^t\overline{Z}_{ij} = \frac{\partial\, {}^tu_i}{\partial\, {}^tx_j} \tag{KI\_17} $$

and call it the *spatial displacement gradient.* Using the same sequence of steps as before, when deriving the Green-Lagrange strain tensor, we rearrange the relation describing the difference of squares of lengths. Without dwelling on details we get

$$ \left(\mathrm{d}^ts\right)^2 - \left(\mathrm{d}^0s\right)^2 = \mathrm{d}^t\mathbf{x}^\mathrm{T}\mathbf{I}\,\mathrm{d}^t\mathbf{x} - \mathrm{d}^t\mathbf{x}^\mathrm{T}\ {}_0^t\mathbf{F}^{-\mathrm{T}}\,{}_0^t\mathbf{F}^{-1}\ \mathrm{d}^t\mathbf{x} = 2\,\mathrm{d}^t\mathbf{x}^\mathrm{T}\underbrace{\frac{1}{2}\left(\mathbf{I} - {}_0^t\mathbf{F}^{-\mathrm{T}}\ {}_0^t\mathbf{F}^{-1}\right)}_{{}_t^t\mathbf{E}^\mathrm{AL}}\mathrm{d}^t\mathbf{x}\,. $$

The notation ${}_0^t\mathbf{F}^{-\mathrm{T}}$ means the inverse of the transpose operation, i.e. $\left({}_0^t\mathbf{F}^{-1}\right)^\mathrm{T}$. If there were no round-off errors then this operation would be equal to $\left({}_0^t\mathbf{F}^\mathrm{T}\right)^{-1}$.

The Almansi strain tensor can be expressed in different forms as well

$$ {}_t^t\mathbf{E}^\mathrm{AL} = \frac{1}{2}\left(\mathbf{I} - {}_0^t\mathbf{F}^{-\mathrm{T}}\,{}_0^t\mathbf{F}^{-1}\right) = \frac{1}{2}\left({}_t^t\overline{\mathbf{Z}} + {}_t^t\overline{\mathbf{Z}}^\mathrm{T} + {}_t^t\overline{\mathbf{Z}}^\mathrm{T}\,{}_t^t\overline{\mathbf{Z}}\right), $$

$$ {}_t^tE_{ij}^\mathrm{AL} = \frac{1}{2}\left(\frac{\partial\, {}^tu_i}{\partial\, {}^tx_j} + \frac{\partial\, {}^tu_j}{\partial\, {}^tx_i} - \frac{\partial\, {}^tu_k}{\partial\, {}^tx_i}\frac{\partial\, {}^tu_k}{\partial\, {}^tx_j}\right). \qquad \ldots \text{(KI\_18)} $$

The Green-Lagrange and Almansi strain tensors describe the same geometrical phenomenon. The former is expressed in coordinates of the reference configuration, i.e. ${}^0x_i$, the latter uses the coordinates of the current configuration, i.e. ${}^tx_i$. Both tensors are independent of the choice of the coordinate system and are invariant with respect to rigid body motions. Their application in solid continuum mechanics is crucial for cases with rigid body displacements and rotations accompanied by finite (not infinitesimal) deformations.

2.7. Cauchy strain tensor – infinitesimal displacements and strains

If the assumptions of small displacements (and rotations), as well as infinitesimal strains, can safely be accepted then the second order terms appearing in Green-Lagrange and Almansi strain tensors expressions can be neglected then both the strain tensors are simplified, are numerically indistinguishable and become to what we call the *Cauchy strain tensor*. Its component contains the derivatives of the first order only, thus the strain is a linear function of displacement increment. On those assumptions, the linear theory of elasticity is based.

To show it, let's differentiate the relation for displacement, i.e. ${}^t u_i = {}^t x_i - {}^0 x_i$, twice. With respect to the Eulerian (spatial) and then with respect to the Lagrangian (material) coordinates. As before, we get

$$\overline{\mathbf{Z}} = \mathbf{I} - \mathbf{F}^{-1} \qquad \text{and} \qquad \mathbf{Z} = \mathbf{F} - \mathbf{I}. \qquad \text{(KI\_19)}$$

The spatial displacement gradient can be rearranged as follows

$$\overline{\mathbf{Z}} = \mathbf{I} - \mathbf{F}^{-1} = \mathbf{I} - \left[\mathbf{I} + \underbrace{(\mathbf{F} - \mathbf{I})}_{\mathbf{Z}}\right]^{-1} = \mathbf{I} - [\mathbf{I} + \mathbf{Z}]^{-1}. \qquad \text{(KI\_20)}$$

Applying the Taylor series expansion for the right-hand side we get

$$\overline{\mathbf{Z}} = \mathbf{I} - [\mathbf{I} - \mathbf{Z} + \mathbf{Z}^2 - \mathbf{Z}^3 + \ldots]$$

and can thus state that both displacement gradients are approximately equal if we can neglect the higher order terms. Thus,

$$\overline{\mathbf{Z}} \cong \mathbf{Z} \qquad \text{or} \qquad \frac{\partial^t u_i}{\partial^t x_j} \cong \frac{\partial^t u_i}{\partial^0 x_j}. \qquad \text{(KI\_21)}$$

Neglecting the higher order increments (derivatives) in Eqs. (KI_10) and (KI_18), we can write

$${}_0^t E_{ij}^{\mathrm{GL}} \cong {}_t^t E_{ij}^{\mathrm{AL}} \cong \frac{1}{2}\left(\frac{\partial^t u_i}{\partial^0 x_j} + \frac{\partial^t u_j}{\partial^0 x_i}\right) \cong \frac{1}{2}\left(\frac{\partial^t u_i}{\partial^t x_j} + \frac{\partial^t u_j}{\partial^t x_i}\right). \qquad \text{(KI\_22)}$$

So, within the realms of the linear theory of elasticity, based on assumptions of infinitesimal displacements and strains, the derivatives of displacements with respect to the reference or to the current coordinates are practically indistinguishable. Then, the Cauchy (infinitesimal) strain tensor is written

$$E_{ij}^{\mathrm{Cauchy}} = \frac{1}{2}\left(\frac{\partial^t u_i}{\partial x_j} + \frac{\partial^t u_j}{\partial x_i}\right). \qquad \text{(KI\_23)}$$

In the linear theory of elasticity, there is no need to distinguish the reference and current coordinates. It practically means that the displacements of a new configuration are calculated, but since they are small, all the consequent analysis is carried out using the initial or reference coordinates, so in the linear theory of elasticity and in the engineering strength of material we as a rule take $x_i \cong {}^t x_i \cong {}^0 x_i$. Also, the upper left index appearing by the displacement quantity is not usually emphasized. Writing the Cauchy infinitesimal strain tensor in full we get

$$
\begin{aligned}
E_{11}^{\text{Cauchy}} &= \frac{\partial u_1}{\partial x_1}, \\
E_{22}^{\text{Cauchy}} &= \frac{\partial u_2}{\partial x_2}, \\
E_{33}^{\text{Cauchy}} &= \frac{\partial u_3}{\partial x_3}, \\
E_{12}^{\text{Cauchy}} = E_{21}^{\text{Cauchy}} &= \frac{1}{2}\left[\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right], \\
E_{23}^{\text{Cauchy}} = E_{32}^{\text{Cauchy}} &= \frac{1}{2}\left[\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right], \\
E_{31}^{\text{Cauchy}} = E_{13}^{\text{Cauchy}} &= \frac{1}{2}\left[\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right].
\end{aligned}
\quad \ldots \text{(KI\_24)}
$$

The Cauchy strain tensor components in the Voigt's notation are

$$
\boldsymbol{\varepsilon}^{\text{Cauchy}} = \begin{Bmatrix} \varepsilon_1^{\text{Cauchy}} \\ \varepsilon_2^{\text{Cauchy}} \\ \varepsilon_3^{\text{Cauchy}} \\ \varepsilon_4^{\text{Cauchy}} \\ \varepsilon_5^{\text{Cauchy}} \\ \varepsilon_6^{\text{Cauchy}} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^{\text{Cauchy}} \\ \varepsilon_{yy}^{\text{Cauchy}} \\ \varepsilon_{zz}^{\text{Cauchy}} \\ \gamma_{xy}^{\text{Cauchy}} \\ \gamma_{yz}^{\text{Cauchy}} \\ \gamma_{zx}^{\text{Cauchy}} \end{Bmatrix} = \begin{Bmatrix} E_{11}^{\text{Cauchy}} \\ E_{22}^{\text{Cauchy}} \\ E_{33}^{\text{Cauchy}} \\ 2E_{12}^{\text{Cauchy}} \\ 2E_{23}^{\text{Cauchy}} \\ 2E_{31}^{\text{Cauchy}} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_1}{\partial x_1} \\ \dfrac{\partial u_2}{\partial x_2} \\ \dfrac{\partial u_3}{\partial x_3} \\ \dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1} \\ \dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2} \\ \dfrac{\partial u_3}{\partial x_1} + \dfrac{\partial u_1}{\partial x_3} \end{Bmatrix}. \quad \text{(KI\_25)}
$$

2.8. Comparison of strain tensors

Example – compare Green-Lagrange, Almansi and Cauchy strain tensors for the longitudinal deformation of a thin rod.

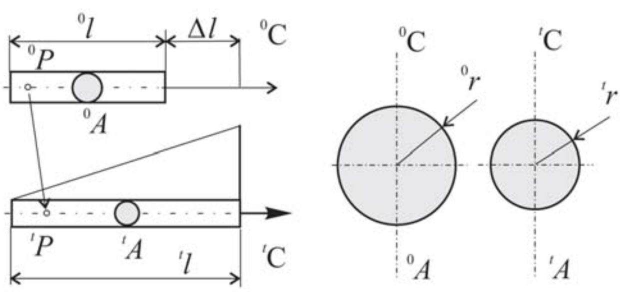


Fig. KI_3 ... Strain rod elongation

A thin prismatic rod of a circular cross-sectional area is clamped at its left end as depicted in Fig. KI_3. In the beginning, in the reference configuration ${}^0C$, the length of the rod is ${}^0l$ and its cross section is ${}^0A$. Due to the deformation, the rod is elongated and narrowed. After the deformation, that is in the current configuration ${}^tC$, the corresponding quantities are ${}^tl$ and ${}^tA$, respectively. The overall increase of the rod's length is $\Delta l = {}^tl - {}^0l$. Similarly, the change of the radius is $\Delta r = {}^tr - {}^0r$. The material particle P, initially positioned at the spatial point ${}^0P$, moves to a new position indicated by ${}^tP$. We assume that the axial displacements of the rod particles are null at the clamping area, and are linearly increasing alongside its length.

Denoting temporally the axial axis of the rod by the lower right index "${}_{\text{ax}}$", the corresponding current axial coordinate can be expressed as a function of the reference coordinate by

$${}^tx_{\text{ax}} = \frac{{}^tl}{{}^0l}\,{}^0x_{\text{ax}} = \frac{{}^0l + \Delta l}{{}^0l}\,{}^0x_{\text{ax}} = \left(1 + \frac{\Delta l}{{}^0l}\right){}^0x_{\text{ax}} = \left(1 + \varepsilon_{\text{ax}}\right){}^0x_{\text{ax}} \quad . \qquad \text{(KI\_26)}$$

The displacements of all material particles in the axial direction are linearly increasing

$$u_{\text{ax}} = {}^tx_{\text{ax}} - {}^0x_{\text{ax}} = \frac{\Delta l}{{}^0l}\,{}^0x_{\text{ax}} = \varepsilon_{\text{ax}}\,{}^0x_{\text{ax}}\,, \qquad \text{(KI\_27)}$$

where we have defined

$$\varepsilon_{\text{ax}} = \frac{\Delta l}{{}^0l}\,. \qquad \text{(KI\_28)}$$

This quantity is called the *linear axial strain*. In the linear theory of elasticity, the adjective *linear* is not emphasized. For the radial dimension – assuming that it follows the same pattern as the axial one, and using the lower right index "${}_{\text{r}}$" – we can write

${}^tr = {}^0r + \Delta r$ and define the linear radial strain as $\varepsilon_r = \dfrac{\Delta r}{{}^0r}$.

Similarly, for radial coordinates

$${}^tx_{\text{r}} = \frac{{}^tr}{{}^0r}\,{}^0x_{\text{r}} = \left(1 + \frac{\Delta r}{{}^0r}\right){}^0x_{\text{r}} = \left(1 + \varepsilon_{\text{r}}\right){}^0x_{\text{r}}\,. \qquad \text{(KI\_29)}$$

Due to the circular cross section of the rod, the ratio of current and reference cross sections is

$$\frac{{}^tA}{{}^0A} = \left(\frac{{}^tr}{{}^0r}\right)^2 = \left(1 + \frac{\Delta r}{{}^0r}\right)^2 = \left(1 + \varepsilon_{\text{r}}\right)^2. \qquad \text{(KI\_30)}$$

For this case of deformation, the generic the transformation relation, i.e. ${}^tx_i = {}^tx_i\left({}^0x_j, t\right)$, has the form

$$
\begin{aligned}
{}^{t}x_1 &= \left(1+\varepsilon_{\mathrm{ax}}\right) {}^{0}x_1, \\
{}^{t}x_2 &= \left(1+\varepsilon_{\mathrm{r}}\right) {}^{0}x_2, \\
{}^{t}x_3 &= \left(1+\varepsilon_{\mathrm{r}}\right) {}^{0}x_3.
\end{aligned}
\qquad \ldots \text{(KI\_31)}
$$

We have assigned the index 1 to the axial direction and indices 2 and 3 to any of radial directions. The material deformation gradient and the Jacobian of the transformation are

$$
{}_{0}^{t}\mathbf{F} = \frac{\partial\, {}^{t}x_i}{\partial\, {}^{0}x_j} = \begin{bmatrix} 1+\varepsilon_{\mathrm{ax}} & 0 & 0 \\ 0 & 1+\varepsilon_{\mathrm{r}} & 0 \\ 0 & 0 & 1+\varepsilon_{\mathrm{r}} \end{bmatrix}, \qquad \text{(KI\_32)}
$$

$$
J = \det {}_{0}^{t}\mathbf{F} = \frac{{}^{t}l}{{}^{0}l}\frac{{}^{t}A}{{}^{0}A}. \qquad \text{(KI\_33)}
$$

Then, the Green-Lagrange strain tensor is

$$
{}_{0}^{t}\mathbf{E}^{\mathrm{GL}} = \frac{1}{2}\left({}_{0}^{t}\mathbf{F}^{\mathrm{T}}\, {}_{0}^{t}\mathbf{F} - \mathbf{I}\right) = \begin{bmatrix} \varepsilon_{\mathrm{ax}} + \varepsilon_{\mathrm{ax}}^2/2 & 0 & 0 \\ 0 & \varepsilon_{\mathrm{r}} + \varepsilon_{\mathrm{r}}^2/2 & 0 \\ 0 & 0 & \varepsilon_{\mathrm{r}} + \varepsilon_{\mathrm{r}}^2/2 \end{bmatrix}. \qquad \text{(KI\_34)}
$$

The strain component ${}_{0}^{t}E_{11}^{\mathrm{GL}}$, corresponding to the axial deformation, could be expressed as

$$
{}_{0}^{t}E_{11}^{\mathrm{GL}} = \frac{\Delta l}{{}^{0}l} + \frac{1}{2}\left(\frac{\Delta l}{{}^{0}l}\right)^2 = \frac{{}^{t}l - {}^{0}l}{{}^{0}l} + \frac{1}{2}\left(\frac{{}^{t}l - {}^{0}l}{{}^{0}l}\right)^2 = \frac{1}{2}\left(\eta^2 - 1\right), \qquad \text{(KI\_35)}
$$

where we have introduced a new dimensionless variable, namely $\eta = {}^{t}l/{}^{0}l$, which is called the *stretch*. Notice, that for the state of no deformation, the stretch is equal to 1.

Similar reasoning, applied to the Almansi strain tensor, allows expressing the axial component in terms of the stretch as follows

$$
{}_{t}^{t}E_{11}^{\mathrm{AL}} = \frac{\Delta l}{{}^{t}l} + \frac{1}{2}\left(\frac{\Delta l}{{}^{t}l}\right)^2 = \frac{{}^{t}l - {}^{0}l}{{}^{t}l} - \frac{1}{2}\left(\frac{{}^{t}l - {}^{0}l}{{}^{t}l}\right)^2 = \frac{1}{2}\left(1 - \frac{1}{\eta^2}\right). \qquad \text{(KI\_36)}
$$

The axial component of the Cauchy strain is obtained by neglecting the second order term in the Green-Lagrange expression, thus

$$
E_{11}^{\mathrm{Cauchy}} = \frac{\Delta l}{{}^{0}l} = \frac{{}^{t}l - {}^{0}l}{{}^{0}l} = \left(\eta - 1\right). \qquad \text{(KI\_37)}
$$

The often used logarithmic strain component could also be considered

$$
E_{11}^{\mathrm{Ln}} = \ln\frac{{}^{t}l}{{}^{0}l} = \ln\eta\,. \qquad \text{(KI\_38)}
$$

Note: Generally, the logarithmic strain (also called natural, true or Hencky) is defined as

$$\mathbf{E}^{\mathrm{LN}} = \frac{1}{2}\ln\left({}_0^t\mathbf{F}^{\mathrm{T}}\,{}_0^t\mathbf{F}\right). \qquad \text{(KI\_39)}$$

See [18].

The axial components with indices "$_{11}$" for all the considered strain tensors, as functions of the stretch variable, are plotted in Fig. KI_4.

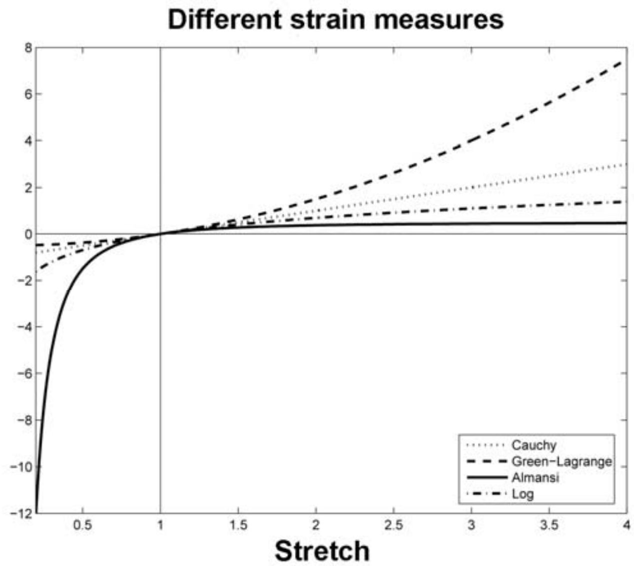


Fig. KI_4 ... Strain components

The limiting values for compressing the rod to the zero length, for $\eta = 0$, or extending it to the infinite length, for $\eta = \infty$, are

Type of strain for	$\eta = 0$	$\eta = \infty$
Green-Lagrange	$-1/2$	∞
Almansi	$-\infty$	$1/2$
Cauchy	-1	∞
Logarithmic	$-\infty$	∞

Observing Fig. KI_4, and the data in the above table, one might be wondering why such a unique geometrical phenomenon, i.e. the rod elongation, is described by so significantly different values of axial strain components.

The different distributions of strain measures should not frighten us. This is the consequence of rather ad hoc definitions of strain measures. Generally, there are infinitely many ways how the strain measures could be defined. Later, we will show that the problem is made unique by a suitable coupling the strain measures with stress measures in such a way that their tensor double dot product gives the mechanical work or energy.

In the vicinity of $\eta = 1$, indicating small or infinitesimal strains, all the considered strain measures are indistinguishable, showing thus that the second order strain measures, as Green-Lagrange and Almansi, need not be worked with. Their applications are, however, imminent for cases with finite deformations and/or for cases with rigid body motions. So, for small displacements and strains (linear theory of elasticity) it is the Cauchy strain tensor that is primarily employed. So,

$$ {}_{t}^{t}E_{ij}^{\mathrm{AL}} \cong {}_{0}^{t}E_{ij}^{\mathrm{GL}} \cong {}_{0}^{t}E_{ij}^{\mathrm{Cauchy}} . \tag{KI\_40}$$

Example – the influence of rigid body motion on the Green-Lagrange and Cauchy strain tensors.

A good example of the rigid body motion is the rotation of a rigid body around a fixed point. In such a case any line segment of that body represents the rotation of the whole body.

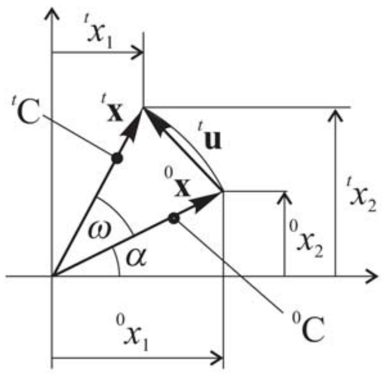


Fig. KI_5 ... Rigid body rotation

In Fig. KI_5 there is shown a 2D case with the material line segment represented by the vector ${}^{0}\mathbf{x}$ belonging to the configuration ${}^{0}C$. This vector rotates with the body to a new configuration ${}^{t}C$ and it is denoted ${}^{t}\mathbf{x}$. Since we are dealing with the rigid body rotation, the lengths of all the material lines do not change and thus $\left|{}^{t}\mathbf{x}\right| = \left|{}^{0}\mathbf{x}\right| = r$. The vector ${}^{t}\mathbf{u} = {}^{t}\mathbf{x} - {}^{0}\mathbf{x}$ represents the displacement of the material point being determined by these vectors.

The displacements components are

$$\begin{aligned} {}^{t}u_1 &= {}^{t}x_1 - {}^{0}x_1 = r\cos(\omega+\alpha) - r\cos(\alpha) = {}^{0}x_1[\cos(\omega)-1] - {}^{0}x_2\sin(\omega), \\ {}^{t}u_2 &= {}^{t}x_2 - {}^{0}x_2 = r\sin(\omega+\alpha) - r\sin(\alpha) = {}^{0}x_1\sin(\omega) + {}^{0}x_2[\cos(\omega)-1]. \end{aligned} \quad \dots \text{(KI\_41)}$$

In this case, the material displacement gradient is

$$ {}_{0}^{t}Z_{ij} = \begin{bmatrix} \dfrac{\partial^{t}u_1}{\partial^{0}x_1} & \dfrac{\partial^{t}u_1}{\partial^{0}x_2} \\ \dfrac{\partial^{t}u_2}{\partial^{0}x_1} & \dfrac{\partial^{t}u_2}{\partial^{0}x_2} \end{bmatrix} = \begin{bmatrix} \cos(\omega)-1 & -\sin(\omega) \\ \sin(\omega) & \cos(\omega)-1 \end{bmatrix}. \tag{KI\_42}$$

Then, the Cauchy and Green-Lagrange strain tensors are

$$\mathbf{E}^{\text{Cauchy}} = \frac{1}{2}\left({}_0^t\mathbf{Z} + {}_0^t\mathbf{Z}^{\mathrm{T}}\right) = \begin{bmatrix} \cos(\omega)-1 & 0 \\ 0 & \cos(\omega)-1 \end{bmatrix}, \tag{KI\_43}$$

$${}_0^t\mathbf{E}^{\text{GL}} = \frac{1}{2}\left({}_0^t\mathbf{Z} + {}_0^t\mathbf{Z}^{\mathrm{T}} + {}_0^t\mathbf{Z}^{\mathrm{T}}\,{}_0^t\mathbf{Z}\right) =$$

$$= \begin{bmatrix} \cos(\omega)-1 & 0 \\ 0 & \cos(\omega)-1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2-2\cos(\omega) & 0 \\ 0 & 2-2\cos(\omega) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{KI\_44}$$

This way, we have shown that The Green-Lagrange strain tensor is really independent of the rigid body rotation, while the Cauchy strain tensor depends on it. Also, we can state that the Cauchy strain tensor might be safely used for infinitesimal values of the angle ω since $\lim\limits_{\omega\to 0}\left[\cos(\omega)-1\right] = 0$.

03_ST. Stress

3.1. What is stress?

The term ***stress*** in current communication is understood differently from the way it is used in mechanical engineering practice. The Cambridge International Dictionary offers for the item ***stress*** the following: *Great worry caused by a difficult situation* or ***a force that acts in a way, which tends to change the shape of an object***[1]. Among many examples from the same source let's quote the one, which might be considered amusing in our engineering community, i.e. *Yoga is a very effective technique for combating stress*. Often the ***stress*** is being considered to be almost equivalent to the **strain** as in: *Many joggers are plagued by knee stress and foot strain caused by unsuitable footwear*. Other sources offer similar explanations. Another example is taken from Wikipedia: *We generally use the word '**stress**' when we feel that everything seems to have become too much - we are overloaded and wonder whether we really can cope with the pressures placed upon us.* ***Anything that poses a challenge or a threat to our well-being is a stress***. *Some stresses get you going and they are good for you - without any stress at all many say our lives would be boring and would probably feel pointless. However, when the stresses undermine both our mental and physical health they are bad. In this text, we shall be focusing on stress that is bad for you.* In our texts, in contradistinction to the previous example, that might invoke a gloomy mood in reader's mind, we will concentrate on meanings **that are good to you**, i.e. on the **mechanical stress** (Spannung in German, contrainte in French, napětí in Czech). The IFToMM (International Federation for the Promotion of Mechanism and Machine Science) online dictionary gives a more acceptable explanation for the ***stress***, i.e.: *Limits of the ratio of force to the area it acts, as the area tends to zero*. The definition of stress, being presented this way, however, says almost nothing about the distribution of the force 'above' the mentioned area. Furthermore, the mentioned dictionary defines the ***stress*** by introducing a new term, namely the ***force*** that is, in turn, specified as an *action*, i.e.: *Action of its surroundings on a body tending to change its state of rest or motion*. Evidently, a definition from the pen of a rigid body person. Other ***force*** definitions appearing in solid mechanics textbooks are not more comprehensive either and describe ***force*** rather circularly by its effects.

A few examples are presented here. In Encyclopedia of Physics – Vol. III/1, on page 532 one finds an alleged d'Alembert's quotation, i.e.: ***Force*** *is only a name for the product of acceleration by mass*.

Similarly, in [18] one finds: ***Forces*** *are vector quantities, which are best described by intuitive concepts as push or pull*.

In terms of proper and clear definitions, the mechanical variables *force* and *stress* can be compared to the definition of *time*. St. Augustine in his Book 11 of Confessions ruminates on the nature of time, asking: *What then is time? If no one asks me, I know: if I wish to explain it to one that asketh, I know not*[2].

So both time and stress (and force and other terms in mechanics, not mentioned here) are consensually defined variables. We understand them rather intuitively; we might have a

[1] The second part of the definition might be agreed with.

[2] *Quid est ergo tempus? Si nemo ex me quaerat, scio; si quaerenti explicare velim, nescio.*

problem to measure them directly, which – however – does not prevent us to purposefully use them in engineering practice. No one would ever have a tendency to challenge them.

A nice definition of stress, from en.wikipedia.org/wiki /Stress_(mechanics), is as follows.

In continuum mechanics, ***stress*** *is a measure of the internal forces acting within a deformable body. Quantitatively, it is a measure of the average force per unit area of a surface within the body on which internal forces act. These internal forces are produced between the particles in the body as a reaction to external forces applied on the body. Because the loaded deformable body is assumed to behave as a continuum, these internal forces are distributed continuously within the volume of the material body, and result in deformation of the body's shape.*

It should, however, be emphasized that the forces in mechanics are of different origins and the loaded area could be related either to the reference or to the current configuration. Generally, we distinguish the *body forces* and the *traction forces*. When the forces are related to the reference configuration then we define so-called *engineering stress*, while the forces related to the current configuration lead to the definition of the *true stress* or the *Cauchy stress*. For small displacements and infinitesimal strains, these two types of stress are numerically indistinguishable. The linear theory of elasticity works with the engineering stress only.

3.2. Body and traction forces

The forces acting on the body from outside are called the *external* or *loading forces*. The forces preventing the body from being torn apart are called the *internal forces*. From outside the internal forces are invisible; to visualize them, we use the technique called the *free body diagram* – we mentally remove a part of the body and replace it by an equivalent system of forces. This way, the internal forces become accessible to the consequent equilibrium analysis. In detail, the principle was thoroughly explained in the text devoted to rigid body mechanics See Fig. ST_1.

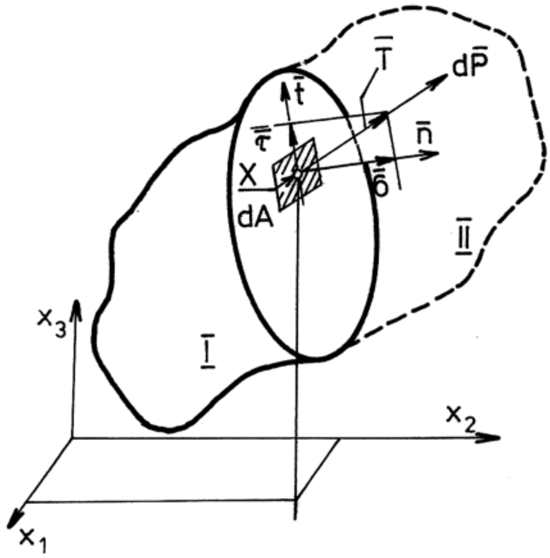


Fig. ST_1 ... Free body diagram

The *external forces* might be classified either as the *body forces* or the *traction forces*.

The **body forces** provide a sort of the action at a distance – they are represented by gravity forces, magnetic forces or by inertia forces. When analyzed, the body forces are related either to a unit of volume – then their dimension is $[\mathrm{N/m^3}]$ and they are sometimes called the *volumetric forces* – or to a unit of mass, then they are measured in $[\mathrm{N/kg}]$.

The notion of body forces emanates from a limit approach of the resultant of elementary forces $\Delta \mathbf{r}_{\mathrm{b}}$ acting on the elementary volume ΔV. The *vector of body forces* is

$$\mathbf{b} = \lim_{\Delta V \to 0} \frac{\Delta \mathbf{r}_{\mathrm{b}}}{\rho \,\mathrm{d}V} . \tag{ST\_1}$$

The **traction forces** act on the surface of the considered body. Their dimension is $\left[\mathrm{N/m^2}\right]$. A typical example is a contact force. If the resultant of elementary forces $\Delta \mathbf{r}_{\mathrm{t}}$ is acting on the elementary area ΔA, then the *vector of traction forces*, often called the *stress vector*[3] is

$$\mathbf{t} = \lim_{\Delta A \to 0} \frac{\Delta \mathbf{r}_{\mathrm{t}}}{\mathrm{d}A} . \tag{ST\_2}$$

The traction forces could also be imagined as acting on the elementary area of the inner part of the body being uncovered due to the process of free-body-diagram reasoning. Then, there is a known relation between the components of the stress vector t_i and the components of the stress tensor T_{ij}. It has the form

$$t_i = T_{ji}\, n_j . \tag{ST\_3}$$

This relation, known as the Cauchy relation, is based on expressing the equivalence of forces acting on elementary material element depicted in Fig. ST_2.

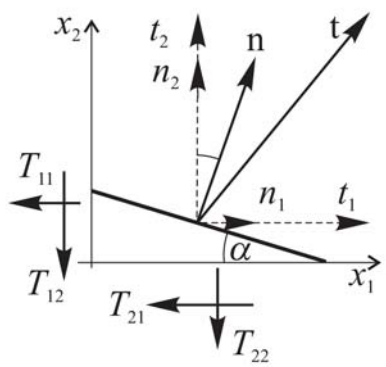


Fig. ST_2 ... Equilibrium forces element

To simplify the derivation of the Cauchy relation, consider a 2D situation, where the equivalence of forces is expressed by two equations.

$$\begin{aligned} t_1\,\mathrm{d}A &= T_{11}\,\mathrm{d}A\sin\alpha + T_{21}\,\mathrm{d}A\cos\alpha, \\ t_2\,\mathrm{d}A &= T_{21}\,\mathrm{d}A\sin\alpha + T_{22}\,\mathrm{d}A\cos\alpha. \end{aligned} \tag{ST\_4}$$

[3] This is a real vector quantity; as such it should be clearly distinguished both from the strain tensor and from the Voigt's stress array.

Since the length of a normal vector $|\mathbf{n}| = 1$, its components thus are

$$n_1 = |\mathbf{n}| \sin\alpha = \sin\alpha, \quad n_2 = |\mathbf{n}| \cos\alpha = \cos\alpha \,. \tag{ST_5}$$

Then,

$$\begin{aligned} t_1 &= T_{11}\, n_1 + T_{21}\, n_2, \\ t_2 &= T_{21}\, n_1 + T_{22}\, n_2. \end{aligned} \tag{ST_6}$$

Generally, in 3D, it holds

$$t_i = T_{ji}\, n_j \qquad \text{or} \qquad t_i = T_{ij}\, n_j \tag{ST_7}$$

since the stress tensor is considered symmetric. The presented analysis was provided in the current configuration and the upper left and upper right indices were temporally omitted.

3.3. True stress and engineering stress

Generally, the stress tensor is considered in the deformed configuration, i.e. ${}^tC$. This tensor is called the *true stress tensor*, or the *Cauchy stress tensor* or simply the *true stress*. It is defined as the ratio of the current forces $\Delta^t \mathbf{r}_\mathrm{t}$ to the geometry of the current configuration, i.e. $\Delta^t A$. In the linear theory of elasticity, the changes of deformation are negligible and are thus neglected. Then, the *engineering stress tensor*, or simply the *engineering stress*, is defined as the ratio of the current forces $\Delta^t \mathbf{r}_\mathrm{t}$, as before, but related to the geometry of the reference configuration. i.e. $\Delta^0 A$.

The true (Cauchy) and engineering stress vectors are thus defined as follows

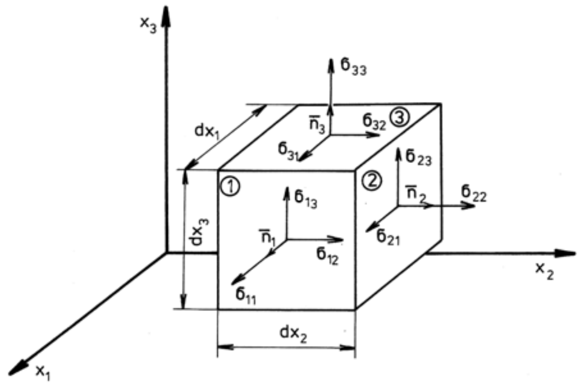
$$\mathbf{t}^{\mathrm{Cauchy}} = \mathbf{t}^{\mathrm{true}} = {}^t_t\mathbf{t} = \lim_{\Delta^t A \to 0} \frac{\Delta^t \mathbf{r}_\mathrm{t}}{\Delta^t A} \qquad \text{and} \qquad \mathbf{t}^{\mathrm{eng}} = {}^t_0\mathbf{t} = \lim_{\Delta^0 A \to 0} \frac{\Delta^t \mathbf{r}_\mathrm{t}}{\Delta^0 A} \,. \tag{ST_8}$$

The relations between the stress vector components, i.e. ${}^t_t t_i$ and ${}^t_0 t_i$, and the stress tensor components, i.e. ${}^t_t T_{ji}$ *and* ${}^t_0 T_{ji}$, for true and engineering quantities, are defined by means of the Cauchy relation, as follows

$$\text{true}: {}^t_t t_i = {}^t_t T_{ji}\ {}^t n_j \ \ldots \text{engineering}: {}^t_0 t_i = {}^t_0 T_{ji}\ {}^0 n_j \,. \tag{ST_9}$$

In the following text, the symbol ${}^t_t T_{ji}$, or in a shortened form T_{ji} , will be used for the true or the Cauchy stress tensor, while the symbol ${}^t_0 T_{ji}$, or in an alternative form ${}^t_0 \Sigma_{ij} = \Sigma_{ij}$, will serve for denoting the engineering stress tensor.

The stress tensor components could be depicted as shown in Fig. ST_3a and Fig. ST_3b.



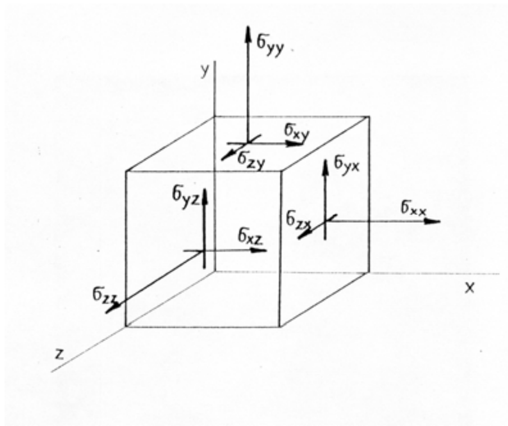


Fig. ST_3a and Fig. ST_3b ... Different notations of 3D stress components

Evidently, the true stress is the correct representation of the state of stress, and the engineering stress is just its approximation. Even if the true stress tensor definition is obvious, its evaluation is from the definition is impossible since the deformed configuration, due to the applied load, is a priory unknown.

In the linear theory of elasticity the displacements are considered small and the strains infinitesimal. Under these conditions, the deformed (current) configuration is negligibly close to the non-deformed (reference) configuration and thus when stresses are evaluated, the initial geometry dimensions ${}^0A$ are used instead of the current dimensions, i.e. ${}^tA$. If the above conditions are accepted, then the true stress becomes numerically indistinguishable from the engineering stress. And since the geometry of the reference configuration is known, the evaluation of the engineering stress is much simpler.

The Cauchy stress is the true measure of the state of stress, while the engineering stress is an acceptable suitable approximation if the above assumptions are satisfied. Care must, however, be taken when non-linear tasks (large deformations and finite strains) are treated – in those cases one has to work with the true stress tensor.

3.4. Motivation for inventing additional stress measures

Inventing so-called fictive stress tensors (i.e. the first and the second Piola-Kirchhoff tensors) allows circumnavigating the problem of the impossibility of the direct true stress evaluation. It should be emphasized that these fictive stress tensors do not have any physical meanings; they represent, however, useful tools for the evaluation of the true stress – the only measure of the state of stress, which is of engineer's interest.

When deriving a 'proper' stress measure we require its independence of rigid body motion and of the choice of the coordinate system.

We will show that both the engineering and true stress tensors do not possess this property.

We will also prove that both Piola-Kirchhoff stress tensors are independent of the rigid body motions and of the choice of the coordinate system, and furthermore that they are energetically conjugate with a suitable choice of the strain measures. By this, it is understood that their tensor double dot product produces the mechanical work or the mechanical energy.

3.5. The first Piola-Kirchhoff stress tensor

Let the elementary force $\mathrm{d}^t\mathbf{r}$ be responsible for the deformation of the elementary tetrahedron from the reference configuration ${}^0C$ to the current configuration ${}^tC$ is depicted in Fig. ST_4.

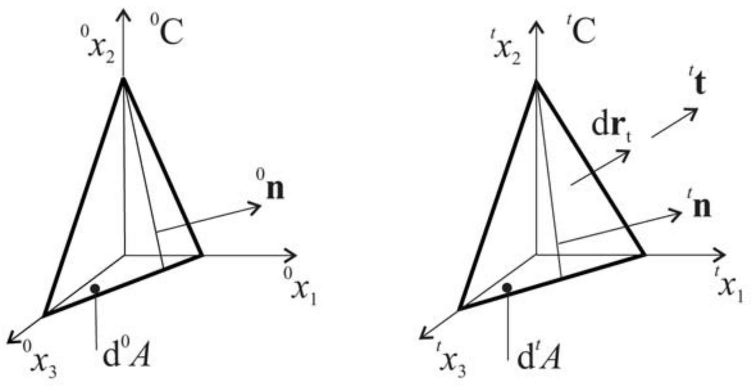


Fig. ST_4 ... Elementary forces tetrahedron

As before, in the current configuration ${}^tC$ the situation is described by the relation between the elementary force and the true stress vector, and by the Cauchy relation, so

$$\mathrm{d}^t\mathbf{r} = {}^t\mathbf{t}\,\mathrm{d}^tA \quad \text{and} \quad {}^t\mathbf{t} = {}^t_t\mathbf{T}^{\mathrm{T}}\,{}^t\mathbf{n}\,. \tag{ST\_10}$$

Eliminating the stress vector from above equations we get

$$\mathrm{d}^t\mathbf{r} = {}^t_t\mathbf{T}\,{}^t\mathbf{t}\,\mathrm{d}^tA\,. \tag{ST\_11}$$

Now, in the configuration ${}^0C$ we 'invent' a fictive force $\mathrm{d}^0\mathbf{r}$ and assume that it is equal to the real force acting in the configuration ${}^tC$[4]. So we suppose that

$$\mathrm{d}^0\mathbf{r} = \mathrm{d}^t\mathbf{r}\,. \tag{ST\_12}$$

The corresponding fictive stress vector ${}^0\mathbf{t}$ is thus defined by

$$\mathrm{d}^0\mathbf{r} = {}^0\mathbf{t}\,\mathrm{d}^0A\,. \tag{ST\_13}$$

To this fictive stress vector ${}^0\mathbf{t}$ there is related a newly defined stress tensor – it is denoted ${}^t_0\mathbf{P}$ and called the *first Piola-Kirchhoff stress tensor*. Again, we use the Cauchy relation

[4] This mental process is a work of fiction, but it safely ends up on the real ground.

$$ {}^{0}\mathbf{t} = {}_{0}^{t}\mathbf{P}^{\mathrm{T}}\ {}^{0}\mathbf{n}\,. \tag{ST\_14} $$

Substituting Eq. (ST_14) into Eq. (ST_13) we get

$$ \mathrm{d}^{0}\mathbf{r} = {}_{0}^{t}\mathbf{P}^{\mathrm{T}}\ {}^{0}\mathbf{n}\ \mathrm{d}^{0}A\,. \tag{ST\_15} $$

Comparing Eq. (ST_15) with Eq. (ST_11) and taking into account Eq. (ST_12) we get the relation between the true stress tensor ${}_{t}^{t}\mathbf{T}$ and the newly defined fictive first Piola-Kirchhoff tensor ${}_{0}^{t}\mathbf{P}$ in the form

$$ {}_{t}^{t}\mathbf{T}^{\mathrm{T}}\ {}^{t}\mathbf{n}\ \mathrm{d}^{t}A = {}_{0}^{t}\mathbf{P}^{\mathrm{T}}\ {}^{0}\mathbf{n}\ \mathrm{d}^{0}A\,. \tag{ST\_16} $$

This is a useful relation, but there are too many unknowns in it so far. To minimize their number we have to determine the relation between the elementary surfaces $\mathrm{d}^{t}A$ and $\mathrm{d}^{0}A$. As before, we rely on the assumption of the mass conservation during the deformation process between configurations ${}^{0}C$ and ${}^{t}C$. So,

$$ {}^{0}\rho\,\mathrm{d}^{0}V = {}^{t}\rho\,\mathrm{d}^{t}V\,. \tag{ST\_17} $$

The initial volume of the un-deformed tetrahedron element in the configuration ${}^{0}C$ is

$$ \mathrm{d}^{0}V = \frac{1}{6}\mathrm{d}^{0}x_{1}\,\mathrm{d}^{0}x_{2}\,\mathrm{d}^{0}x_{3} \tag{ST\_18} $$

and can be rearranged into the form

$$ \mathrm{d}^{0}V = \frac{1}{9}\left(\mathrm{d}^{0}a_{1}\,\mathrm{d}^{0}x_{1} + \mathrm{d}^{0}a_{2}\,\mathrm{d}^{0}x_{2} + \mathrm{d}^{0}a_{2}\,\mathrm{d}^{0}x_{2}\right) = \frac{1}{9}\left(\mathrm{d}^{0}a_{i}\,\mathrm{d}^{0}x_{i}\right) = \frac{1}{9}\mathrm{d}^{0}\mathbf{a}^{\mathrm{T}}\mathrm{d}^{0}\mathbf{x}\,, \tag{ST\_19} $$

where we have introduced a new 'vector' variable containing the projections of the elementary area $\mathrm{d}^{0}A$ into the coordinate plates in the form

$$ \mathrm{d}^{0}\mathbf{a} = \begin{Bmatrix} \mathrm{d}^{0}a_{1} \\ \mathrm{d}^{0}a_{2} \\ \mathrm{d}^{0}a_{3} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}\mathrm{d}^{0}x_{2}\,\mathrm{d}^{0}x_{3} \\ \frac{1}{2}\mathrm{d}^{0}x_{3}\,\mathrm{d}^{0}x_{1} \\ \frac{1}{2}\mathrm{d}^{0}x_{1}\,\mathrm{d}^{0}x_{2} \end{Bmatrix}. \tag{ST\_20} $$

So, the relation between the area $\mathrm{d}^{0}A$ and its projections could be expressed by means of the normal vectors, as

$$ \mathrm{d}^{0}\mathbf{a} = \mathrm{d}^{0}A\ {}^{0}\mathbf{n} \qquad \text{or} \qquad \mathrm{d}^{0}A = \mathrm{d}^{0}\mathbf{a}^{\mathrm{T}}\ {}^{0}\mathbf{n} = {}^{0}\mathbf{n}^{\mathrm{T}}\ \mathrm{d}^{0}\mathbf{a}\,. \tag{ST\_21} $$

Analogically, the volume of the elementary tetrahedron in the configuration ${}^{t}C$ is

$$ \mathrm{d}^{t}V = \frac{1}{6}\mathrm{d}^{t}x_{1}\,\mathrm{d}^{t}x_{2}\,\mathrm{d}^{t}x_{3} = \frac{1}{9}\mathrm{d}^{t}\mathbf{a}^{\mathrm{T}}\,\mathrm{d}^{t}\mathbf{x}\,. \tag{ST\_22} $$

Substituting Eqs. (ST_22), (ST_19) into Eq. (ST_17) we get

$$ {}^{0}\rho \, \mathrm{d}^{0}\mathbf{a}^{\mathrm{T}} \, \mathrm{d}^{0}\mathrm{x} = {}^{t}\rho \, \mathrm{d}^{\mathrm{t}}\mathbf{a}^{\mathrm{T}} \, \mathrm{d}^{\mathrm{t}}\mathrm{x} \, . \tag{ST\_23} $$

Expressing the current coordinates $\mathrm{d}^{\mathrm{t}}\mathbf{x}$ as functions of the reference coordinates $\mathrm{d}^{0}\mathbf{x}$ by means of the deformation gradient we get

$$ \mathrm{d}^{t}\mathbf{x} = {}^{t}_{0}\mathbf{F} \, \mathrm{d}^{0}\mathbf{x} \, . \tag{ST\_24} $$

Thus,

$$ {}^{0}\rho \mathrm{d}^{0}\mathbf{a}^{\mathrm{T}} \, \mathrm{d}^{0}\mathrm{x} = {}^{t}\rho \mathrm{d}^{\mathrm{t}}\mathbf{a}^{\mathrm{T}} \, {}^{0}_{t}\mathbf{F} \, \mathrm{d}^{0}\mathrm{x} \, . $$

The previous expression has to be independent of the choice of the coordinate system, so it simplifies to

$$ {}^{0}\rho \mathrm{d}^{0}\mathbf{a}^{\mathrm{T}} = {}^{t}\rho \mathrm{d}^{t}\mathbf{a}^{\mathrm{T}} \, {}^{0}_{t}\mathbf{F} \quad . \tag{ST\_25} $$

Using Eq. (ST_21), the Eq. (ST_25) can be rearranged into

$$ \mathrm{d}^{0}A \, {}^{0}\mathbf{n}^{\mathrm{T}} = \frac{{}^{t}\rho}{{}^{0}\rho} \mathrm{d}^{\mathrm{t}}A \, \mathbf{n}^{\mathrm{T}} \, {}^{t}_{0}\mathbf{F} \, . \tag{ST\_26} $$

Substituting Eq. (ST_26) into Eq. (ST_16) we get

$$ {}^{t}_{t}\mathbf{T}^{\mathrm{T}} \, \mathrm{d}^{t}A \, {}^{t}\mathbf{n}^{\mathrm{T}} = \frac{{}^{t}\rho}{{}^{0}\rho} \, {}^{t}_{0}\mathbf{P}^{\mathrm{T}} \, {}^{t}_{0}\mathbf{F}^{\mathrm{T}} \, \mathrm{d}^{\mathrm{t}}A \, {}^{t}\mathbf{n} \, . \tag{ST\_27} $$

The result has to be independent of the choice of the elementary area defined by its normal, so the relation between the true stress tensor and the first Piola-Kirchhoff stress tensor is

$$ {}^{t}_{t}\mathbf{T} = \frac{{}^{t}\rho}{{}^{0}\rho} \, {}^{t}_{0}\mathbf{F} \, {}^{t}_{0}\mathbf{P} \quad . \tag{ST\_28} $$

The inverse relation, expressing the Piola-Kirchhoff stress tensor as a function of the true stress tensor, is obvious

$$ {}^{t}_{0}\mathbf{P} = \frac{{}^{0}\rho}{{}^{t}\rho} \, {}^{t}_{0}\mathbf{F}^{-1} \, {}^{t}_{t}\mathbf{T} \, . \tag{ST\_29} $$

We have shown that the ratio of densities before and after the deformation could be expressed as the Jacobian of the transformation ${}^{t}\mathbf{x} = {}^{t}\mathbf{x}\left({}^{0}\mathbf{x}, t\right)$ defined between configurations ${}^{0}C$ and ${}^{t}C$, i.e.

$$ J = \frac{{}^{0}\rho}{{}^{t}\rho} = \det\left({}^{t}_{0}\mathbf{F}\right), \tag{ST\_30} $$

so Eq. (ST_29) could be rewritten into the form

$$ {}_0^t\mathbf{P} = J \; {}_0^t\mathbf{F}^{-1} \; {}_t^t\mathbf{T} . \tag{ST\_31}$$

It is obvious that the product of the non-symmetric deformation gradient and the symmetric true stress gives a non-symmetric result. So, the first Piola-Kirchhoff stress tensor is non-symmetric.

3.5. The second Piola-Kirchhoff stress tensor

The non-symmetry of the first Piola-Kirchhoff stress tensor is an unpleasant feature and leads to further considerations. To derive a new – this time symmetric stress tensor – let's 'invent' an alternative fictive force acting in the configuration ${}^0C$. Instead of accepting $\mathrm{d}^0\mathbf{r} = \mathrm{d}^t\mathbf{r}$, as before, we define

$$ \mathrm{d}^0\mathbf{r} = {}_0^t\mathbf{F}^{-1} \; \mathrm{d}^t\mathbf{r} . \tag{ST\_32}$$

This relation for forces is based on the analogy of the previous relation for coordinates that was derived in the form $\mathrm{d}^0\mathbf{x} = {}_0^t\mathbf{F}^{-1} \; \mathrm{d}^t\mathbf{x}$.

Following the same sequence of steps as before, when deriving the first Piola-Kirchhoff stress tensor, we arrive at the expression relating the true stress ${}_t^t\mathbf{T}$ and the second Piola-Kirchhoff, say ${}_0^t\mathbf{S}$, in the form

$$ {}_t^t\mathbf{T} = \frac{{}^t\rho}{{}^0\rho} \; {}_0^t\mathbf{F} \; {}_0^t\mathbf{S} \; {}_0^t\mathbf{T} . \tag{ST\_33}$$

The inverse relation is

$$ {}_0^t\mathbf{S} = \frac{{}^0\rho}{{}^t\rho} \; {}_0^t\mathbf{F}^{-1} \; {}_t^t\mathbf{T} \; {}_0^t\mathbf{F}^{-\mathrm{T}} . \tag{ST\_34}$$

On the right-hand side of Eq. (ST_34) one can see the quadratic form of variables, so the second Piola-Kirchhoff is really symmetric.

3.6. Piola-Kirchhoff stress tensors in the linear theory of elasticity

It should be emphasized that both the first and the second Piola-Kirchhoff stress tensors have little physical meaning. They represent, however, useful tools for the treatment geometrically non-linear tasks.

In the linear theory of elasticity the change of volume and thus the change of density due to the deformation is neglected, so the Jacobian of the transformation $J = \dfrac{{}^0\rho}{{}^t\rho} = 1$. Furthermore, the deformation gradient $\mathbf{F} = \mathbf{I}$, since the current coordinates are approximately considered to be identical with the reference coordinates and thus the first Piola-Kirchhoff and the second Piola-Kirchhoff and the true stress tensors are approximately equal to engineering stress tensor. And as said before, the true stress tensor under conditions of small deformations and infinitesimal strains becomes the engineering stress tensor.

So, in the linear theory of elasticity, we approximately take that

$$ {}_0^tP_{ij} \cong {}_0^tS_{ij} \cong {}_t^tT_{ij} \cong {}_0^t\Sigma_{ij} \,. \qquad \text{(ST\_35)} $$

The Green-Lagrange strain tensor, multiplied by the second Piola-Kirchhoff stress tensor – by means of the double dot product, i.e. $\frac{1}{2}\,{}_0^tE_{ij}\;{}_0^tS_{ij}$ – gives the scalar quantity which represents the mechanical energy or the mechanical work.

For more details [7], [12a], [14], [18], [19], [23], [25], [28], [36].

04_CR. Constitutive models

4.1. Material models

After explaining and defining the geometry of deformation – gauged by strain measures – occurring due to loading – quantified by stress measures – it is time to pose the question of the relation between the strain and stress measures. Such a relation is often called the *constitutive relation*. Having it at our disposal allows determining the 'amount' of the deformation due to the prescribed loading. And together with the theory of the strength of material and the failure theories to determine the ability of a machine part, made of a particular material, to withstand the particular loading.

There are thousands of engineering materials and they deform differently when being loaded. To ascertain their geometric response to the prescribed loading, a properly prepared experiment is needed. The experimental results, in the form of *stress-strain relation*, have to be generalized and presented in a suitable mathematical form to be used in engineering computations.

In this text, we will concentrate on the mathematical models describing the stress-stress behaviour in bodies made of different materials. These models are of phenomenological nature, they disregard the actual corpuscular structure of materials – they describe the material as being continuous, homogeneous, with no gaps. It should be emphasized that each model has the clear limits of its validity but does not contain embedded warnings about its misuse. It is always the analyst who is fully responsible for the application of the model within proper limits of its applicability.

The material models can roughly be classified as follows

- Linear elastic
- Nonlinear elastic
- Hyperelastic
- Hypoelastic
- Elastoplastic
- Creep
- Viscoplastics

For more details see [6], [7], [15], [17], [18], [21], [39].

4.2. Linear elastic material

In this text, we will primarily limit our attention to the linear elastic model. For the detailed study of other material models, listed above, the book [6] is recommended.

The linear elastic model is based on the validity of Hooke's law $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ or $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$, which means that the infinitesimal (Cauchy) strain is linearly proportional to the engineering stress. The tensor C_{ijkl} and the matrix $\mathbf{C}_{6\times 6}$ represent the proportionality 'constant'.

The Hooke's law is an acceptable approximation of the material behaviour under the following assumptions.

- There is a unique relationship between stress and strain.
- Material properties are independent of the specimen size.
- Strains are said to be reversible, it means that no hysteresis occurs.
- The material is the rate-of-loading independent.
- No thermodynamic effects are considered.
- The material is homogeneous, which means that $\lim \Delta m = \lim \rho \Delta V = \rho \lim \Delta V$.
- Corpuscular structure of the matter is disregarded.
- Generally, it is valid for the fully anisotropic material behaviour.

In mechanical engineering, the linear elastic material model is the most frequently used model for the material behaviour. It is applicable for cases with small displacements and infinitesimal strains. This model assumes that the material deformation (expressed in infinitesimal strains) linearly depends on the applied loading (expressed in engineering stress) – it is known as the *generalized Hooke's law*. Its tensor and Voigt's forms are

$$\Sigma_{ij} = C_{ijkl}\, E_{kl} \qquad \text{and} \qquad \sigma_i = C_{ij}\, \varepsilon_j \qquad \text{or} \qquad \boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}. \qquad \text{(CR\_1)}$$

The tensor C_{ijkl} and the corresponding matrix $\mathbf{C}_{6\times6}$ represent the proportionality 'constant'. Sometimes, the matrix $\mathbf{C}_{6\times6}$ is called the *matrix of elastic moduli*.

The stress tensor and the Voigt's stress array are presented here in various forms appearing in textbooks.

$$\boldsymbol{\Sigma} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{bmatrix} \qquad \dots \qquad \boldsymbol{\sigma} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{Bmatrix} \Sigma_{11} \\ \Sigma_{22} \\ \Sigma_{33} \\ \Sigma_{12} \\ \Sigma_{23} \\ \Sigma_{31} \end{Bmatrix} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_z \\ \tau_x \\ \tau_y \end{Bmatrix}. \qquad \text{(CR\_2)}$$

The Cauchy infinitesimal strain tensor and the Voigt's strain array could be written as

$$\mathbf{E} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \qquad \dots \qquad \boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{Bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{12} \\ 2E_{23} \\ 2E_{31} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}. \qquad \text{(CR\_3)}$$

The tensor C_{ijkl}, appearing in the tensor form of the generalized Hooke's law, is symmetric and has 81 components. Generally, there are 21 independent material constants for the linear, homogeneous and fully un-isotropic material. See [18]. The equivalent matrix $\mathbf{C}_{6\times6}$, appearing in the Voigt's notation, has 36 components. It is symmetric as well.

In the case of the *linear isotropic material*, there are only 2 independent material constants, i.e. the Young modulus E and the Poisson ratio υ.

The simplest un-isotropic material behaviour is known as the *orthotropic* – it is characterized by different material properties in two mutually perpendicular directions and in this case there are 9 independent material constants. This behaviour is typical for materials with warp-and-weft structures, for rolled sheet steel plates, for wood, etc. The survey of more complicated un-isotropic materials can be found in [6].

The simplest material model is described by the following attributes – linear, anisotropic and homogeneous[1]. How to find the material constants for this kind of materials is briefly sketched in the following paragraphs.

The story of stress and strain measures presented in paragraphs devoted to strain and stress is retold here, this time using the engineering style based on treating the individual cases sequentially, starting from the simplest and proceeding to more complicated ones. In this part of the text, dealing with the linear theory of elasticity, we will exclusively work with engineering stresses and infinitesimal (Cauchy) strains, without repeatedly specifying this fact. For practical engineering computations, the Voigt's notation is almost exclusively used.

4.3. Uniaxial stress

This is the simplest loading mode. All the applied loading forces are acting within a single line of action. The loading of an elementary element, representing this loading mode, is shown in Fig. CR_1. It is assumed that the direction of loading is associated with the x direction.

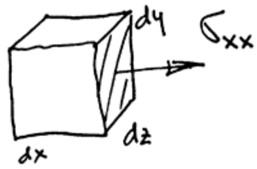


Fig. CR_1 ... 1D stress

Example. If a prismatic rod of a constant cross-sectional area A is loaded by an axial force F, then the axial stress is $\sigma = \dfrac{F}{A}$.

4.3.1. Prelude – Uniaxial state of stress expressed by equivalent stress components in an oblique cross section

A clamped thin rod, see Fig. CR_2, with the cross-sectional area S_0 is loaded by an axial force F.

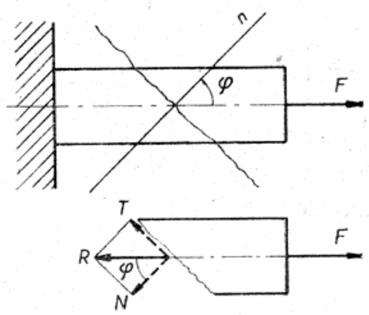


Fig. CR_2 ... Stress components in oblique cross section

The corresponding axial stress is $\sigma_0 = F / S_0$. Let's cut the rod by a fictional plane defined by the normal n forming an angle φ with the lateral axis, say x.

[1] Linear means that the strain is proportional to stress, isotropic means the material properties in different directions are identical and homogeneous means that the material properties within the body are the same. The last attribute also means that the corpuscular structure of the matter is neglected.

Consequently, the cut cross section has the area of $S = S_0 / \cos\varphi$. Using the free-body-diagram principle, we add an internal force R, representing the removed part of the body, that has to be in equilibrium with the loading force F. So, $R = F$.

Then, the internal force is decomposed into the normal and tangential components as depicted in Fig. CR_2. So,

$$N = R\cos\varphi; \quad T = R\sin\varphi. \tag{CR\_4}$$

The normal stress (corresponding to the normal force) in the oblique cross-section is

$$\sigma = \frac{N}{S} = \frac{R\cos\varphi}{S_0/\cos\varphi} = \frac{R\cos^2\varphi}{S_0} = \frac{F}{S_0}\cos^2\varphi, \tag{CR\_5}$$

while the tangential stress (corresponding to the tangential force) in the oblique cross-section is

$$\tau = \frac{T}{S} = \frac{R\sin\varphi}{S_0/\cos\varphi} = \frac{F}{S_0}\sin\varphi\,\cos\varphi$$

.

... (CR_6)

It should be emphasized that the actual state of stress in the body is unique, it does not change. We have only expressed it by equivalent stress components in differently inclined planes characterized by the varying angle φ.

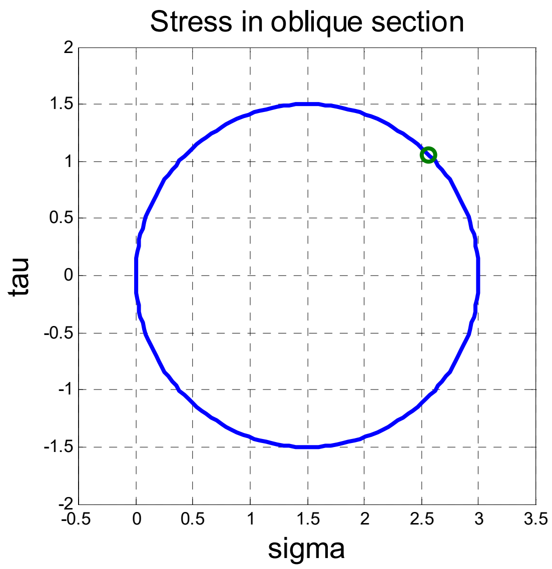


Fig. CR_3 ... Stress components in oblique cross section

Using the trigonometric relations for the double angular arguments and reminding that $\sigma_0 = F / S_0$, we get

$$\sigma = \frac{1}{2}\sigma_0(1+\cos 2\varphi), \tag{CR\_7}$$

$$\tau = \frac{1}{2}\sigma_0 \sin 2\varphi. \tag{CR\_8}$$

In the coordinate system (σ, τ), these relations represent the parametric equations of a circle having the radius $\frac{1}{2}\sigma_0$. Both stress components, as functions of the angle φ, are computed by the program `mpp_005e_oblique_section_c1` and are depicted in Fig. CR_3.

```
% mpp_005e_oblique_section_c1
clear
f = 0:1:180; fi = pi*f/180;
sig0 = 3;
sig = sig0*0.5*(1 + cos(2*fi)); tau = sig0*0.5*sin(2*fi);
f22 = 22.5;
fi22 = pi*f22/180;
sig22 = sig0*0.5*(1 + cos(2*fi22)); tau22 = sig0*0.5*sin(2*fi22);
figure(1)
plot(sig,tau, sig22,tau22,'o','linewidth' ,2.5, 'markersize', 8)
axis('equal'); % axis('tight')
axis([-0.5 3.5 -2 2])
title('Stress in oblique section', 'fontsize', 16)
xlabel('sigma', 'fontsize', 16)
ylabel('tau', 'fontsize', 16)
grid
```

4.4. Plane stress and plane strain

This loading mode is characterized by the fact that all the loading forces are applied within a single plane. The loading of an elementary cube, representing this loading mode, is depicted in Fig. CR_4.

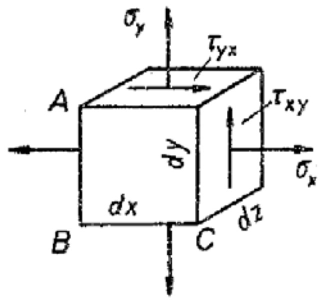


Fig. CR_4 ... 2D stress

Depending on how the elementary cube is constrained we distinguish two cases.

If the face ABC of the cube is free to deform in the *z*-axis direction, then we are dealing with so-called *plane stress state of stress*.

If the face ABC of the cube – together with its parallel face – are restrained (no displacements in that direction allowed) then we have the case called *plane strain state of stress*. In detail, we will analyze both mentioned cases later.

By the term *plane state of stress* we understand the idealized situation when a body is principally loaded in a certain plane only, and thus the forces and stresses in the direction perpendicular to that plane are considered to be equal to zero. See Fig. CR_4 and Fig. CR_5.

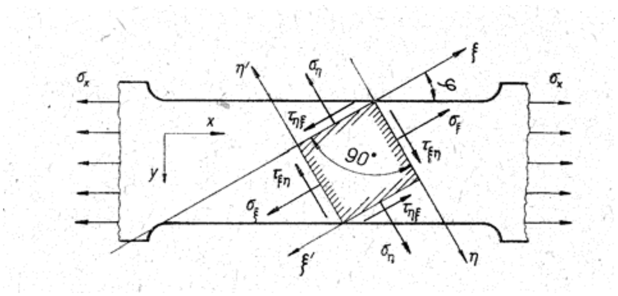


Fig. CR_5 ... A loaded strip

Beforehand, it should be emphasized that the corresponding deformations and strains in that perpendicular directions are non-zero because the specimen is not constrained in that direction and is thus allowed to ‘breathe’ freely.

Instead of rods, we have dealt with so far, consider a thin strip, depicted in Fig. CR_5, whose transverse dimension, say h, in the direction z, perpendicular to the plane of the drawing x, y, is relatively small with respect to the strip dimension in that plane. It is assumed that the distribution of the axial stress within the transverse cross-sectional area is uniform. Assume that in the direction of x- axis, the strip is loaded by the prescribed stress σ_x defined in the coordinate system x, y. Let's analyze what are stress components in another coordinate system[2] defined by axes ξ, η. The ξ axis forms an angle φ with the x axis.

4.5. Transformation of stress components into another coordinate system

Applying the same reasoning as before and using a slightly different notation, we get the relations for the *normal* and *tangential* components of stress in the coordinate system ξ, η. The newly defined stress components are called the *normal stress* and the *shear stress*, respectively.

$$\sigma_{\xi\xi} = \sigma_{\xi} = \frac{1}{2}\sigma_x(1+\cos 2\varphi),$$
$$\sigma_{\xi\eta} = \tau_{\xi\eta} = \frac{1}{2}\sigma_x \sin 2\varphi. \qquad \ldots \text{(CR\_9)}$$

Now, define a new coordinate system, say ξ', η', that is turned counterclockwise with respect to the system ξ, η by an angle π as depicted in Fig. CR_5. Due to the periodicity of trigonometric functions we get

$$\sigma_{\eta} = \frac{1}{2}\sigma_x(1+\cos(2\varphi+\pi)) = \frac{1}{2}\sigma_x(1-\cos 2\varphi),$$
$$\tau_{\eta\xi} = \frac{1}{2}\sigma_x(\sin(2\varphi+\pi)) = -\frac{1}{2}\sigma_x \sin 2\varphi. \qquad \ldots \text{(CR\_10)}$$

Notice, that the absolute values of normal and shear stresses differ by a sign only. Analogical results can be derived for opposite coordinate systems that were obtained by the rotations of $\pi/2$ and $3\pi/2$ respectively with respect to the system ξ, η.

Let's extend the present analysis by assuming that the strip is loaded not only by the stress σ_x but also by the stress σ_y in the perpendicular direction, defined by the y-coordinate.

Then, the considered element, shown in Fig. CR_6, will have the internal normal stresses (σ_x, σ_y) and internal shear stresses (τ_{xy}, τ_{yx}) components acting at all of its sides.

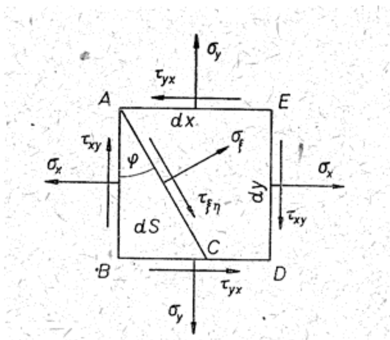


Fig. CR_6 ... Plane stress

Since the element is infinitesimal – its dimensions are $\mathrm{d}x \times \mathrm{d}y$ – we disregard the changes in stress quantities alongside the element dimensions.

From the theory of rigid body mechanics, it is known, that three conditions have to be met to satisfy the equilibrium of a body in the plane. Two of them – in directions of x and y – are evidently satisfied identically. The forces acting in x-and y-directions are equal but of

[2] It should be emphasized again that the state of stress at a given point is still the same – only its stress components differ.

opposite signs. The remaining equilibrium condition is of the moment type. The shear stresses acting at the element sides are τ_{xy}, τ_{yx}. For the corresponding shear forces (stress × area × distance), the moment equation, related to the center of the element, has the form

$$\tau_{xy} h \mathrm{d}y \frac{\mathrm{d}x}{2} - \tau_{yx} h \mathrm{d}x \frac{\mathrm{d}y}{2} = 0 . \qquad \text{(CR\_11)}$$

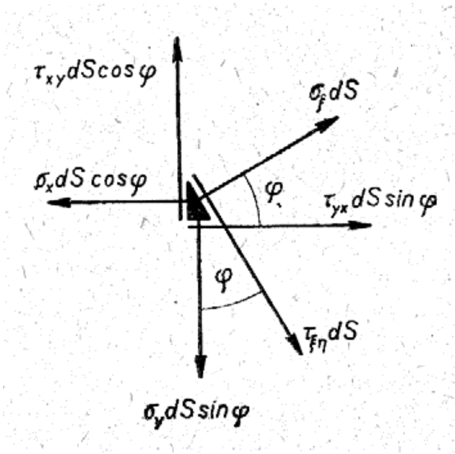
From this follows that

$$\tau_{xy} = \tau_{yx} . \qquad \text{(CR\_12)}$$

This identity expresses the so-called *rule of conjugate shear stresses*[3].

4.6. Mohr's circle representation for two-dimensional state of stress

For the given state of stress, characterized by the stress components $\sigma_x, \sigma_y, \tau_{xz}$ in the basic coordinate system, we have to determine the corresponding stress components, say $\sigma_\xi, \tau_{\xi\eta}$, in the cross section AC, defined by the normal line inclined by the angle φ with respect the x - axis.



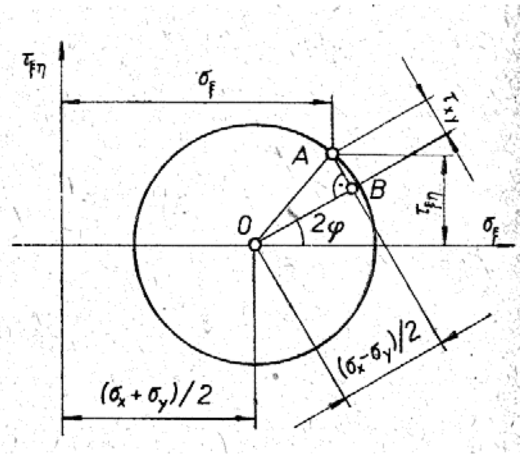


Fig. CR_7 ... Equilibrium of forces acting on ABC element.

Fig. CR_8 ... Corresponding Mohr's circle.

Consider the equilibrium of forces acting on the element ABC, depicted in Fig. CR_6 and in Fig. CR_7. The cross-sectional area corresponding to the line AC is $\mathrm{d}S$, and its projections are $\mathrm{d}S \sin\varphi$ and $\mathrm{d}S \cos\varphi$ respectively.

Taking the material element as the point in a plane, only two scalar equilibrium equations are needed.

---

[3] This is not a law – it is an accepted rule. All this reasoning is based on assumptions of neglecting infinitesimal increments of higher orders. There is a so called Cosserat theory of continuum which takes into account the increments of higher orders. See COSSERAT E., COSSERAT F. *Théorie des corps déformables,* Hermann, pp. iii-xlv, 2009. In that theory the *rule of conjugate shear stresses* is not valid.

$$\xi: \quad \sigma_\xi \mathrm{d}S - \sigma_x \mathrm{d}S\cos^2\varphi - \sigma_y \mathrm{d}S\sin^2\varphi + \tau_{xy}\mathrm{d}S\cos\varphi\sin\varphi + \tau_{yx}\mathrm{d}S\sin\varphi\cos\varphi = 0,$$
$$\eta: \quad \tau_{\xi\eta}\mathrm{d}S - \sigma_x \mathrm{d}S\cos\varphi\sin\varphi + \sigma_y \mathrm{d}S\sin\varphi\cos\varphi - \tau_{xy}\mathrm{d}S\cos^2\varphi + \tau_{yx}\mathrm{d}S\sin^2\varphi = 0. \quad \ldots \text{(CR\_13)}$$

Realizing that $dS = h\mathrm{d}y/\cos\varphi = h\mathrm{d}x/\sin\varphi$ and that $\tau_{xy} = \tau_{yx}$ we get

$$\sigma_\xi = \sigma_x\cos^2\varphi + \sigma_y\sin^2\varphi - 2\tau_{xy}\sin\varphi\cos\varphi,$$
$$\tau_{\xi\eta} = (\sigma_x - \sigma_y)\cos\varphi\sin\varphi + \tau_{yx}(\cos^2\varphi - \sin^2\varphi) = 0. \quad \ldots \text{(CR\_14)}$$

Using the double-argument relations for trigonometric functions, i.e.

$$\sin 2\varphi = 2\sin\varphi\cos\varphi \quad a \quad \cos 2\varphi = \cos^2\varphi - \sin^2\varphi = 1 - \sin^2\varphi = 2\cos^2\varphi - 1, \quad \text{(CR\_15)}$$

and rearranging we get

$$\sigma_\xi = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\varphi - \tau_{xy}\sin 2\varphi,$$
$$\tau_{\xi\eta} = \frac{\sigma_x - \sigma_y}{2}\sin 2\varphi + \tau_{xy}\cos 2\varphi. \quad \ldots \text{(CR\_16)}$$

These equations represent a circle in $\sigma_\xi, \tau_{\xi\eta}$ coordinates. It is called the *Mohr's circle* – it depicts a graphical representation of the stress components in different planes, defined by a varying angle φ. It should be emphasized again, that the same state of stress at a point of a loaded body could be expressed by different stress components belonging to particular cross-sectional directions defined by the angle φ. The situation is depicted in Fig. CR_8. Due to the rearrangement of the above formulas by the double-argument relations, we use the angle 2φ in the Mohr's circle diagram instead of the actual 'material' angle φ.

4.7. Drawing the Mohr's circle with rule and compasses

for the stress components $\sigma_x, \sigma_y, \tau_{xy}$ measured in the coordinate system x, y.

- Draw the Cartesian coordinate system with σ, τ axes.
- Plot two points A ($\sigma_x, -\tau_{xy}$) and B(σ_y, τ_{xy}).
- To do so, we have to accept a sign convention for shear stress components in the 'material space' and in the 'Mohr's circle space'. In literature, there are a few approaches. In the text, we will follow the procedure depicted in Fig. CR_9. In the material space, the shear stress components could have the appearance shown in subplots 1a and 2a respectively. This kind of shear stress (being applied separately) would evoke the deformations depicted in subplots 1c and 2c respectively. For the purposes of a unique representation in the 'Mohr's circle space', we accept the following rule. If the outer normal $\vec{n}$, being turned clockwise by the angle $\pi/2$, coincides with the direction of the considered shear stress component, see 2c, then that shear stress component is plotted as a positive value in the Mohr's circle diagram, see 2d. Otherwise, the counterclockwise rotation of the normal, see 1b, requires plotting that component as the negative value in the Mohr's diagram as indicated in 1e.

- Another step. Connecting the points A and B we get the point S from which we can draw a circle with the radius $r = \overline{SA}$.

4.8. Finding principal stresses by means of the Mohr's circle

The circle intersects the σ - axis at points 1 and 2. At these two points, there are no shear stress components. The corresponding stress components σ_1 and σ_2 are so-called *principal stresses*. They are important for evaluating the strength of material capabilities. We will deal with the subject in more detail later.

Observing the Mohr's circle we might notice and define:

- the radius $r = \sqrt{\left(\frac{1}{2}\left(\sigma_x - \sigma_y\right)^2 + \tau_{xy}^2\right)}$,
- principal stresses $\sigma_{1,2} = \dfrac{\sigma_x + \sigma_y}{2} \pm \sqrt{\dfrac{(\sigma_x - \sigma_y)^2}{2} + \tau_{xy}^2}$,
- average stress $\sigma_{\text{avg}} = \frac{1}{2}(\sigma_x + \sigma_y)$,
- maximum stress $\sigma_{\max} = \sigma_1 = \sigma_{\text{avg}} + r$,
- minimum stress $\sigma_{\min} = \sigma_2 = \sigma_{\text{avg}} - r$,
- maximum and minimum shear stress $\tau_{\max,\min} = \pm r$,
- plane orientation for the principal stress σ_1 is obtained from $\tan 2\varphi = \dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}$, while the corresponding material angle is φ. It is an oriented angle and the corresponding orientations are indicated in subplots 1e and 2e.

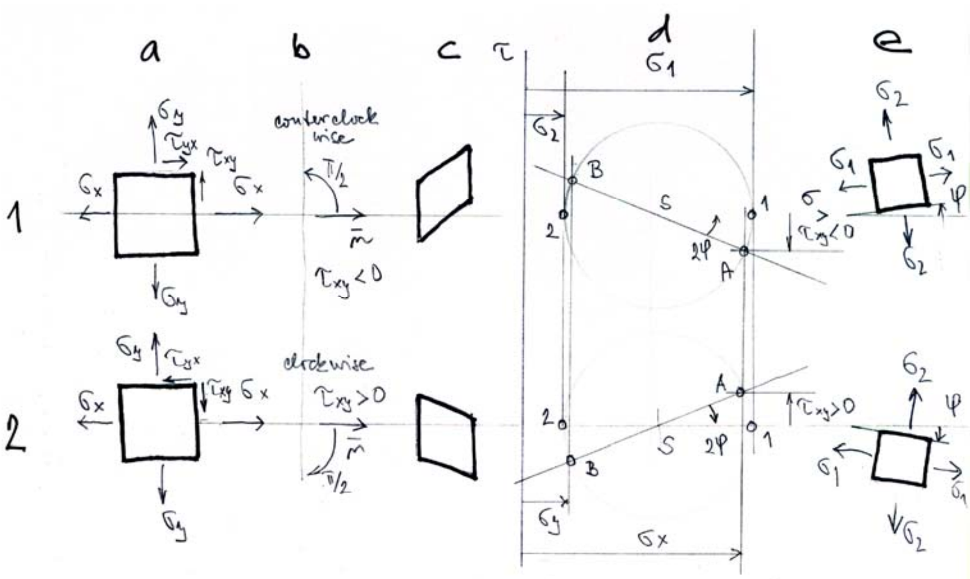


Fig. CR_9 ... Mohr's circle and sign convention.

4.9. Finding principal stresses by means of the standard eigenvalue problem

Let's remind how the same vector could be expressed in two Cartesian coordinate systems x, y and ξ, η, having the same origin, and being turned by the angle φ. See Fig. CR_10.

The scalar notation of this transformation is

$$\begin{aligned} A_\xi &= A_x \cos\varphi + A_y \sin\varphi \\ A_\eta &= -A_x \sin\varphi + A_y \cos\varphi \end{aligned}, \qquad \ldots \text{(CR\_17)}$$

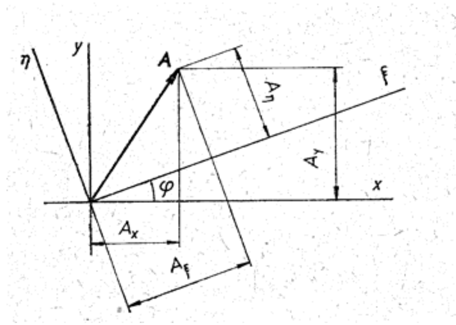


Fig. CR_10 ... Vector in two coordinate systems

while the equivalent matrix expression, using the rotation matrix $\mathbf{R}$, is

$$\begin{Bmatrix} A_\xi \\ A_\eta \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}}_{\mathbf{R}} \begin{Bmatrix} A_x \\ A_y \end{Bmatrix} = \mathbf{R} \begin{Bmatrix} A_x \\ A_y \end{Bmatrix}. \qquad \text{(CR\_18)},$$

The same expression in an alternative notation, where the letters denoting the axes are replaced by numbers such as $x \to 1,\ y \to 2$, leads to

$$\begin{Bmatrix} A_1' \\ A_2' \end{Bmatrix} = \mathbf{R} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}; \quad \mathbf{a}' = \mathbf{R}\mathbf{a}. \qquad \text{(CR\_19)}$$

Let's introduce a new entity, namely the *stress vector*, which expresses the force acting in a particular cross-section determined by a unit normal, as depicted in Fig. CR_11. The stress vector, denoted $\mathrm{d}\mathbf{F}$, has components $\mathrm{d}F_x, \mathrm{d}F_y$, while the corresponding normal can be expressed as

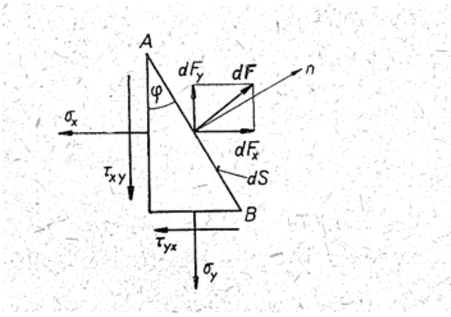


Fig. CR_11 ... Stress vector and stress components

$$\vec{n} = \mathbf{n} = \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} \cos\varphi \\ \sin\varphi \end{Bmatrix}. \qquad \text{(CR\_20)}$$

The condition of equivalence of elementary forces in the x - direction requires that

$$\mathrm{d}F_x = \sigma_x \,\mathrm{d}S \cos\varphi + \tau_{xy} \,\mathrm{d}S \sin\varphi .$$

Similarly for the y - direction. The stress components are defined as elementary forces related to a unit of corresponding areas. So, the stress vector could be written as

$$\vec{f} = \mathbf{f} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{Bmatrix} \mathrm{d}F_x / \mathrm{d}S \\ \mathrm{d}F_y / \mathrm{d}S \end{Bmatrix} = \begin{Bmatrix} \sigma_x \cos\varphi + \tau_{xy} \sin\varphi \\ \sigma_y \sin\varphi + \tau_{xy} \cos\varphi \end{Bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix} \begin{Bmatrix} \cos\varphi \\ \sin\varphi \end{Bmatrix} = \underbrace{\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}}_{\mathbf{\Sigma}} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \mathbf{\Sigma n}\,.$$

... (CR_21)

We have already taken into consideration the validity of the *rule of conjugate shear stresses*, i.e. $\sigma_{12} = \sigma_{21}$. Using the tensor notation, we could write

$$f_i = \sigma_{ij} n_j\,. \qquad \text{(CR\_22)}$$

This expression represents two equations for 2D ($i = 1,2$) and three equations ($i = 1,2,3$) for 3D space. So, Eq. (CR_21) in a plane could be rewritten into

$$f_i = \sum_{j=1}^{2} \sigma_{ij} n_j; \quad i = 1,2\,. \qquad \text{(CR\_23)}$$

This is the so-called *Cauchy relation* which relates the stress vector components f_i to the stress components σ_{ij} by means of the normal components n_j. See [29], [32], [35], [36].

4.10. Principal stress

It should be emphasized that the notion of the principal stress plays an important role in continuum mechanics. The principal stress is a scalar quantity, uniquely attached to the stress tensor quantity – generally composed of nine components. The principle stress is a tool allowing express the “magnitude” of the tensor quantity and permits to determine the safe conditions for a material to withstand the applied loading.

Let’s find such a cross-section, defined by an angle φ, in which the stress vector $\mathbf{f}$ becomes a λ-multiple of the normal line $\mathbf{n}$. So, we require that the relation $f_i = \lambda n_i$ holds. In other words, we require that the stress vector has the same direction as the normal vector defining the cross-sectional area at which the stress vector ‘lives’. This could happen if and only if the matrix $\mathbf{\Sigma}$ becomes diagonal. Under such conditions, the shear stresses vanish. Mathematically, this is the matrix eigenvalue problem which leads to the diagonalization of a matrix.

Substituting the required condition $f_i = \lambda n_i$ into Eq. (CR_22) we get

$$\begin{aligned} \sigma_{ij} n_j - \lambda n_i &= 0, \\ \sigma_{ij} n_j - \lambda \delta_{ij} n_j &= 0. \end{aligned} \qquad \text{… (CR\_24)}$$

Here, we have used the Kronecker’s symbol δ_{ij}, for which $\delta_{ij} = \begin{cases} 1 & \text{pro } i = j \\ 0 & \text{pro } i \neq j \end{cases}$. (CR_25)

This allows expressing the normal components as $n_i = \delta_{ij} n_j$, or $\begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix}$. (CR_26)

In the matrix form, we have

$$\left[\begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right]\begin{Bmatrix}n_1\\ n_2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}, \tag{CR\_27}$$

while in a scalar notation we can write

$$\begin{aligned}(\sigma_{11} - \lambda)n_1 + \sigma_{12}n_2 = 0,\\ \sigma_{12}n_1 + (\sigma_{22} - \lambda)n_2 = 0.\end{aligned} \tag{CR\_28}$$

This is a system of homogeneous algebraic equations. They have zero on the right-hand side. Such a system has a non-trivial solution only if its determinant is equal to zero. Thus,

$$\begin{vmatrix}\sigma_{11} - \lambda & \sigma_{12}\\ \sigma_{12} & \sigma_{22} - \lambda\end{vmatrix} = 0. \tag{CR\_29}$$

The determinant evaluation leads to the quadratic equations whose roots are

$$\lambda_{1,2} = \frac{\sigma_{11} + \sigma_{22}}{2} \pm \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}. \tag{CR\_30}$$

The roots λ_1 and λ_2 are eigenvalues of the matrix $\mathbf{\Sigma} = \begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{bmatrix}$. Physically, they represent the principal stresses. They are denoted by σ_1 and σ_2 respectively. Thus, as before we get

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\frac{(\sigma_x - \sigma_y)^2}{2} + \tau_{xy}^2}. \tag{CR\_31}$$

The state of stress at each point of a loaded body is defined by a unique stress tensor formed by an array of nine stress components. The set of these components differs, depending on the orientation of the coordinate system in which the stress components are expressed. Using different but currently used notations we might express the stress tensor σ_{ij} as

$$\sigma = \begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{bmatrix} = \begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}\end{bmatrix} = \begin{bmatrix}\sigma_{xx} & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \sigma_{yy} & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \sigma_{zz}\end{bmatrix} = \begin{bmatrix}\sigma_x & \tau_z & \tau_y\\ \tau_z & \sigma_y & \tau_x\\ \tau_y & \tau_x & \sigma_z\end{bmatrix}. \tag{CR\_32}$$

This tensor is symmetric since we have accepted the rule of conjugate shear stress components, i.e. $\sigma_{ij} = \sigma_{ji}$.

This is a form frequently used in texts devoted to the mathematical theory of elasticity. In engineering, we prefer another notation[4] based on the above-mentioned symmetry. From it follows that only six stress components – out of nine – are independent. Thus, it suffices to

[4] Sometimes called the the Voigt's notation.

express the state of stress as an array of non-repeated stress components. They are usually assembled in such a way that the normal stress components are listed first, being followed by the shear stress components.

$$\{\sigma_{\mathrm{Voigt}}\} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_z \\ \tau_x \\ \tau_y \end{Bmatrix}. \tag{CR\_33}$$

Example – determine the principal stresses graphically and numerically

Given: The state of stress is given by the stress components $\sigma_x = 50; \sigma_y = -10; \tau_z = \tau_{xy} = 40$. All in [MPa]. See Fig. CR_12.

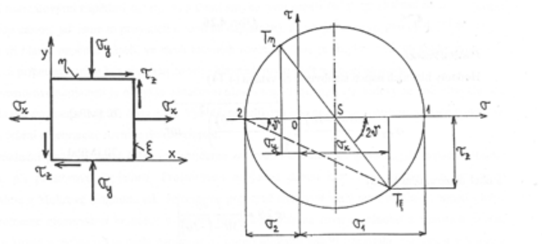


Fig. CR_12 ... 2D State of stress.

Determine: The principal stresses and their directions. Use the graphical and the numerical approach.

Defining the proper scale of drawing, using the rule and compasses, considering the sign convention and applying the procedure described above, we obtain the values of principal stresses as $\sigma_1 = 70\,\mathrm{Mpa}, \sigma_2 = -30\,\mathrm{Mpa}$. Using the formula $\tan 2\vartheta = \dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}$, we get $\vartheta = 26°30'$.

And now, numerically – using the **program mpp_008e_principal_stress_c1**.

```
% mpp_008e_principal_stress_c1
clear
% stress components
sx = 50; sy = -10; txy = 40;

% stress matrix
sig = [sx txy; txy sy];

% find eigenvectors and eigenvalues
[v, lambda] = eig(sig);

% components of the first eigenvector are the actual normals
n1 = v(1,1); n2 = v(2,1);

% the angle between normals in radiand and degrees
psi = atan(n2/n1);
psi_deg = 180*psi/pi + 90

% eigenvalues of the stress matrix are the principal stresses
s1 = lambda(1,1)
s2 = lambda(2,2)
```

Running the program we get

```
psi_deg = 26.5651;    s1 = -30;     s2  = 70;
```

Since a picture is worth a thousand words, study carefully Fig. CR_13 to understand the plane state of stress.

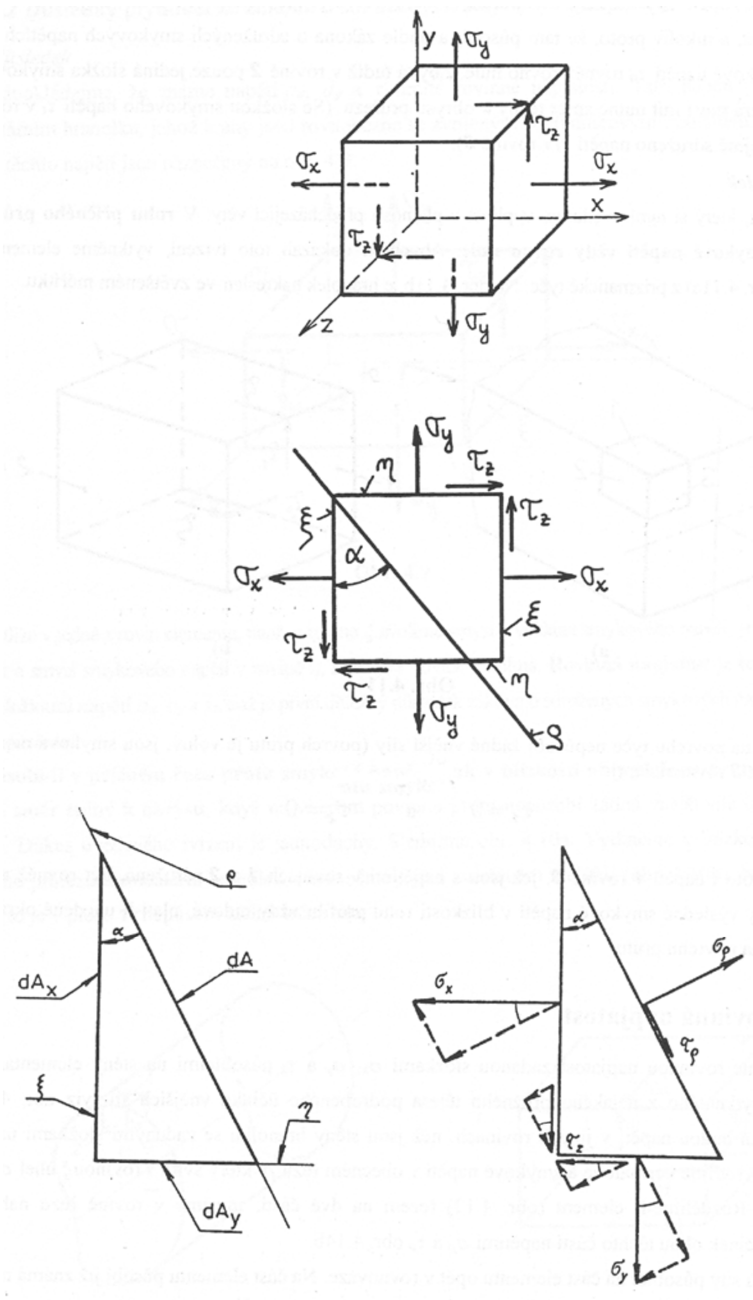


Fig. CR_13 ... Understanding the plane state stress

4.11. Plane state of strain

If a prismatic rod of a constant cross-sectional area of the length l is loaded by an axial force, then the overall axial deformation is Δl, and the corresponding axial strain is $\varepsilon = \dfrac{\Delta l}{l}$.

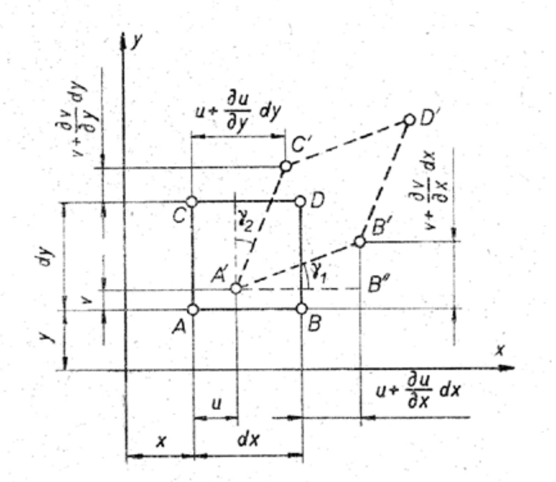


Fig. CR_14 ... Plane strain deformation

Consider the deformation of an initially square element shown in Fig. CR_14.

Assume that due to the prescribed deformation a material point, initially located in A, moves to A'. Its displacement, decomposed into directions of coordinate axes x, y are denoted u and v, respectively.

To express the displacement components of the point B, we utilize the Taylor series and neglect the quantities of the second and higher orders. We assume that the functions describing the displacement field, together with their derivatives, are smooth. Then, for the horizontal and vertical displacement components of the point B, one can write.

$$u_{\mathrm{B}} = u(x + \mathrm{d}x, y) = u(x, y) + \left.\frac{\partial u}{\partial x}\right|_{x,y} \mathrm{d}x + \ldots \approx u + \frac{\partial u}{\partial x}\mathrm{d}x\,, \tag{CR\_34a}$$

$$v_{\mathrm{B}} = v(x + \mathrm{d}x, y) = v(x, y) + \left.\frac{\partial v}{\partial x}\right|_{x,y} \mathrm{d}x + \ldots \approx v + \frac{\partial v}{\delta x}\mathrm{d}x\,. \tag{CR\_34b}$$

Denoting the length of AB $= \mathrm{d}x$, then the length of A'B' is

$$\mathrm{A'B'} = \sqrt{(\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}x)^2 + (\frac{\partial v}{\partial x}\mathrm{d}x)^2}\,. \tag{CR\_35}$$

The relative change of the considered line in the x- direction is called the strain and is expressed as

$$\varepsilon_x = \frac{\mathrm{A'B'{-}AB}}{\mathrm{AB}} = \frac{\sqrt{(\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}x)^2 + (\frac{\partial v}{\partial x}\mathrm{d}x)^2} - \mathrm{d}x}{\mathrm{d}x} = \sqrt{\frac{(\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}x)^2 + (\frac{\partial v}{\partial x}\mathrm{d}x)^2}{\mathrm{d}x^2}} - 1 =$$

$$= \sqrt{\frac{dx^2 + 2dx\frac{\partial u}{\partial x}\mathrm{d}x + (\frac{\partial u}{\partial x}\mathrm{d}x)^2 + (\frac{\partial v}{\partial x}\mathrm{d}x)^2}{dx^2}} - 1 = \sqrt{1 + 2\frac{\partial u}{\partial x} + (\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial x})^2} - 1 =$$

$$= 1 + \frac{1}{2}(2\frac{\partial u}{\partial x} + (\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial x})^2) - 1 = \frac{\partial u}{\partial x} + \tfrac{1}{2}(\frac{\partial u}{\partial x})^2 + \tfrac{1}{2}(\frac{\partial v}{\partial x})^2 . \qquad \text{(CR\_36)}$$

When simplifying the above formula we have used an approximation for expressing the square root function as $\sqrt{1+x} \cong 1 + x/2$. This approximation is safely applicable only if the x value is substantially less than 1. It does not hurt to observe the order of that approximation. See the Matlab program **square_root_approximation.m**.

```
% square_root_approximation.m
format long e
i = 0;
xrange = [0.1 0.01 0.001 0.0001 0.000001];
for x = xrange
    i = i + 1;
    a1 = sqrt(1 + x);
    a2 = 1 + x/2;
    rel = (a1 - a2)/a1;
    r(i,:) = [a1 a2 rel];
end
rr = [xrange' r]
```

The program output is

x	sqrt(1+x)	1+x/2	rel. error
1e-01	1.048808848170152e+00	1.050000000000000e+00	-1.135718707871894e-03
1e-02	1.004987562112089e+00	1.005000000000000e+00	-1.237616103905176e-05
1e-03	1.000499875062461e+00	1.000500000000000e+00	-1.248751170242068e-07
1e-04	1.000049998750062e+00	1.000050000000000e+00	-1.249875216692911e-09
1e-06	1.000000499999875e+00	1.000000500000000e+00	-1.250110500672832e-13

Similarly, for the y - direction.

Generally, the longitudinal strains for plane deformation are

$$\varepsilon_x = \frac{\partial u}{\partial x} + \tfrac{1}{2}(\frac{\partial u}{\partial x})^2 + \tfrac{1}{2}(\frac{\partial v}{\partial x})^2 \approx \frac{\partial u}{\partial x}, \qquad \text{(CR\_37a)}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} + \tfrac{1}{2}(\frac{\partial v}{\partial y})^2 + \tfrac{1}{2}(\frac{\partial v}{\partial y})^2 \approx \frac{\partial v}{\partial y}. \qquad \text{(CR\_37b)}$$

In the linear theory of elasticity, the quadratic terms are neglected.

And now, the *shear deformation*. See Fig. CR_14 again. Due to the deformation, the initially right angle BAC is changed by

$$\gamma = \gamma_1 + \gamma_2 , \qquad \text{(CR\_38)}$$

It is known that

$$\sin\gamma = \sin(\gamma_1 + \gamma_2) = \sin\gamma_1 \cos\gamma_2 + \cos\gamma_1 \sin\gamma_2 . \qquad \text{(CR\_39)}$$

Also

$$\sin\gamma_1 = \frac{\frac{\partial v}{\partial x}\mathrm{d}x}{\mathrm{A'B'}}, \qquad \sin\gamma_2 = \frac{\frac{\partial u}{\partial y}\mathrm{d}y}{\mathrm{A'C'}}, \tag{CR\_40}$$

$$\cos\gamma_1 = \frac{\mathrm{d}x + \frac{\partial u}{\mathrm{d}x}\mathrm{d}x}{\mathrm{A'B'}}, \qquad \cos\gamma_2 = \frac{\mathrm{d}y + \frac{\partial v}{\mathrm{d}y}\mathrm{d}y}{\mathrm{A'C'}}. \tag{CR\_41}$$

Substituting Eqs. (CR_40), (CR_41) into Eq. (CR_39) we get

$$\sin\gamma = \frac{\mathrm{d}x\mathrm{d}y}{(\mathrm{A'B'})(\mathrm{A'C'})}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right]. \tag{CR\_42}$$

For small angles, i.e. $\gamma \to 0$, we could approximate $\sin\gamma \approx \tan\gamma \approx \gamma$ and $\cos\gamma \to 1$.

The area of the non-deformed element is $\mathrm{d}x\mathrm{d}y$, while that of the deformed one is

$$(\mathrm{A'B'})(\mathrm{A'C'})\sin(\frac{\pi}{2} - \gamma) = (\mathrm{A'B'})(\mathrm{A'C'})\cos\gamma\,. \tag{CR\_43}$$

By common consent, the shear strain is defined as the tangent of the angle γ multiplied by the ratio of deformed and non-deformed areas.

Generally, the shear strain is

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}, \tag{CR\_44}$$

while for small deformations the relation is simplified by neglecting the quantities of higher orders. Then, it has the form

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{CR\_45}$$

Concluding, the strain components – for the plane deformation case – are defined by

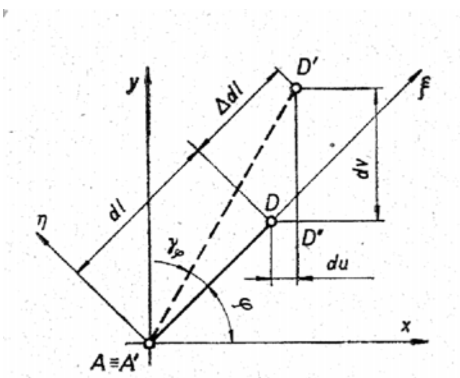
$$\varepsilon_x = \frac{\partial u}{\partial x}, \qquad \varepsilon_y = \frac{\partial v}{\partial y}, \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{CR\_46}$$

These relations represent so-called kinematic relations expressing the strains as functions of displacements.

Often, these relations are formulated by means of the differential operators as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}. \tag{CR\_47}$$

4.12. Transformation of strain components into another coordinate system



In the previous paragraph, we analyzed how the undeformed length of AD diagonal, depicted in Fig. CR_14, changed – due to the applied deformation – into the line having the length A′D′. Using simple geometric considerations, we derived the strain components $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ in the coordinate system x, y.

Fig. CR_15 ... Transformation of strain components

Now, we are interested in how these strain components change when expressed in a different coordinate system ξ, η which is rotated, with respect to the original one, by the angle φ as depicted in Fig. CR_15. It should be emphasized that the strain quantity is still the same – it is independent of the chosen coordinate system – its strain components, however, differ.

Projecting the increments of displacements $\mathrm{d}u, \mathrm{d}v$ of the point D into the ξ direction we get

$$\Delta \mathrm{d}l = \mathrm{d}u \cos\varphi + \mathrm{d}v \sin\varphi . \tag{CR\_48}$$

The change of direction of observed lines (for small deformations) could be expressed by

$$\gamma_\varphi \approx \tan\gamma_\varphi = \frac{-\mathrm{d}u \sin\varphi + \mathrm{d}v \cos\varphi}{\mathrm{d}l}. \tag{CR\_49}$$

Assuming that the displacements $u = u(x,y), v = v(x,y)$ are continuous functions of coordinates, their increments can be expressed in the form

$$\mathrm{d}u = \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y \ \text{ and } \ \mathrm{d}v = \frac{\partial v}{\partial x}\mathrm{d}x + \frac{\partial v}{\partial y}\mathrm{d}y . \tag{CR\_50}$$

Observing Fig. CR_15 one can write

$$\mathrm{d}x = \mathrm{d}l\cos\varphi \text{ a } \mathrm{d}y = \mathrm{d}l\sin\varphi . \tag{CR\_51}$$

Then, the strain in the direction of the ξ coordinate is

$$\varepsilon_\xi = \frac{\Delta \mathrm{d}l}{\mathrm{d}l} = \frac{\mathrm{d}u\cos\varphi + \mathrm{d}v\sin\varphi}{dl} = \frac{\left(\frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y\right)\cos\varphi + \left(\frac{\partial v}{\partial x}\mathrm{d}x + \frac{\partial v}{\partial y}\mathrm{d}y\right)\sin\varphi}{dl} =$$

$$= \left(\frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}l} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}l}\right)\cos\varphi + \left(\frac{\partial v}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}l} + \frac{\partial v}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}l}\right)\sin\varphi =$$

$$= \left(\frac{\partial u}{\partial x}\cos\varphi + \frac{\partial u}{\partial y}\sin\varphi\right)\cos\varphi + \left(\frac{\partial v}{\partial x}\cos\varphi + \frac{\partial v}{\partial y}\sin\varphi\right)\sin\varphi =$$

$$= \left(\frac{\partial u}{\partial x}\cos^2\varphi + \frac{\partial u}{\partial y}\sin\varphi\cos\varphi\right) + \left(\frac{\partial v}{\partial x}\sin\varphi\cos\varphi + \frac{\partial v}{\partial y}\sin^2\varphi\right). \tag{CR\_52}$$

So,

$$\varepsilon_\xi = \frac{\partial u}{\partial x}\cos^2\varphi + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\sin\varphi\cos\varphi + \frac{\partial v}{\partial y}\sin^2\varphi . \tag{CR\_53}$$

Similarly for the η direction. From

$$\gamma_\varphi = \frac{-\mathrm{d}u\sin\varphi + \mathrm{d}v\cos\varphi}{\mathrm{d}l} \tag{CR\_54}$$

we get

$$\gamma_\varphi = -\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\sin\varphi\cos\varphi - \frac{\partial u}{\partial y}\sin^2\varphi + \frac{\partial v}{\partial x}\cos^2\varphi \tag{CR\_55}$$

and

$$\gamma_{\varphi+\frac{\pi}{2}} = -\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\sin\varphi\cos\varphi - \frac{\partial u}{\partial y}\cos^2\varphi + \frac{\partial v}{\partial x}\sin^2\varphi . \tag{CR\_56}$$

The shear component is given by the difference of angles corresponding to ξ,η directions

$$\gamma_{\xi\eta} = 2\left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)\sin\varphi\cos\varphi + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\left(\cos^2\varphi - \sin^2\varphi\right). \tag{CR\_57}$$

So finally, the strain components are

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{CR\_58}$$

Substituting the Eqs. (CR_53), (CR_57) into the previous one we get the shear strain component for the ξ- direction being turned by the angle φ with respect to the original coordinate system

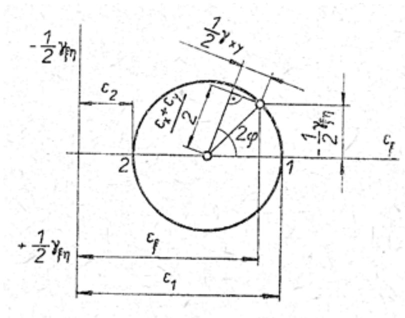
$$\varepsilon_\xi = \varepsilon_x \cos^2\varphi + \gamma_{xy}\sin\varphi\cos\varphi + \varepsilon_y \sin^2\varphi, \tag{CR\_59a}$$

$$\gamma_{\xi\eta} = 2(\varepsilon_y - \varepsilon_x)\sin\varphi\cos\varphi + \gamma_{xy}(\cos^2\varphi - \sin^2\varphi). \tag{CR\_59b}$$

Using the trigonometric relations for the double argument we get

$$\begin{aligned} \varepsilon_\xi &= \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\varphi + \frac{1}{2}\gamma_{xy}\sin 2\varphi, \\ \frac{1}{2}\gamma_{\xi\eta} &= -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\varphi + \frac{1}{2}\gamma_{xy}\cos 2\varphi. \end{aligned} \quad \text{… (CR\_59, c,d)}$$

These relations are formally similar to those derived for stress components in Eq. (CR_60). So, the similar conclusions could be deduced for the strain components related to various directions and a corresponding Mohr's circle for strain components could be plotted. Fig. CR_16 graphically represents relations in Eq. (CR_60). Also, the principal strains are clearly defined.



The actual deformation of bodies is always three dimensional. To simplify things we have assumed that the analyzed body is approximately two dimensional and has the shape of a thin strip, whose outer faces are parallel with the (x, y) plane and whose transversal dimension is negligible. Furthermore, we pretended that in the z direction there is no deformation. This is, however, a crude, on the other hand often useful, idealization of reality.

Fig. CR_16 ... Mohr's circle for strains

To describe the state of idealized two-dimensional deformations, we generally proceed by two independent ways.

Either we neglect the stress components occurring in directions perpendicular to the z-axis. This is a model called the *plane stress*. The stress components $\sigma_x, \sigma_y, \tau_{xy}$ are non-zero and other components, i.e. $\sigma_z, \tau_{zx}, \tau_{zy}$, are neglected.

Or we assume that the strain components $\varepsilon_z = \gamma_{zx} = \gamma_{zy} = 0$. This model is called the *plane strain* or *plane deformation*.

4.13. Tension (tensile) testing

Before presenting and analyzing the constitutive relations for a linear isotropic material, let's briefly dwell into experimental procedures needed to determine the material constants. It is a mature and extensive subject of material science requiring usually a full semester course. Here, only a brief survey is presented. The thermal effects and the 'speed' of loadings are not analyzed.

It is the *tension test*, which is a basic tool for finding the fundamental material constants that are needed for the strength-of-material computations. The test is unique for each material. A specimen, see Fig. CR_17, is subjected to successively increasing values of axial forces until the failure.

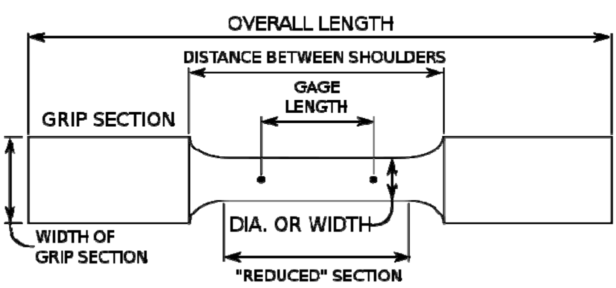


Fig. CR_17 ... Specimen for tensile test

During the test, the axial force as a function of axial elongation is registered. Then, the data are recomputed into the axial stress vs. axial strain quantities. The resulting plot is known as the *stress-strain diagram*. Schematically, it is depicted in Fig. CR_18.

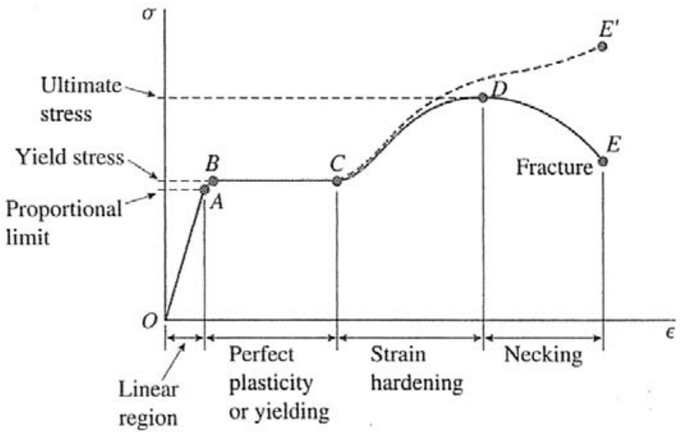


Fig. CR_18 ... Stress-strain diagram

The *stress-strain diagram*, seen in Fig. CR_18, is highly idealized. It describes the axial (longitudinal) stress σ as a function of the axial strain ε. The function starts at the point O corresponding to initial conditions – that is no stress, no strain. Then, due to the gradually increasing axial loading, the stress-strain function rises linearly to the point A that is called the *proportionality limit*. In this region, the stress is directly proportional to the strain – we say that in this region the material obeys the *Hooke's law* – for 1D cases could be expressed in the scalar form

$$\sigma = E\varepsilon \,. \qquad \text{(CR\_60)}$$

The coefficient of proportionality E is called *Young's modulus*. The stress corresponding to the point A is called the *proportional limit*, say σ_{A}. As a rule, it is defined by the occurrence of the linearity deviation not-exceeding the value of 0.005 %.

The term *linearity* should be clearly distinguished from the term *elasticity*. By elasticity, we understand the ability of the loaded body – after it is unloaded – to return to its original state. Generally, the elastic material might follow a non-linear stress-strain curve – what is in this case important is that the material – when being unloaded – does not show permanent deformations – in other words, is not subjected to the plasticity behaviour. So, the *elasticity* is the ability of a material to return to its previous shape after the loading stress is released – regardless of the linear or non-linear loading stress-strain behaviour. The non-linear but still elastic behaviour is typical for rubber materials.

If the specimen is subjected to a still increased load, we come to the point B. From now on, the structural bindings of material start to collapse and the *permanent deformations* occur. When the material is unloaded we witness the phenomenon called *hysteresis* – the body will not return to its original geometrical shape.

The stress corresponding to the point B, i.e. σ_{Y}, is called the *yield stress*. When the loading stress overcomes this value, the internal permanent deformations lead to so-called *yielding* of the material or by other words to almost *perfect plasticity*. In this loading region, the strain increases without a noticeable increase of stress. The *yield strength* indicates the crucial situation where permanent deformations of material occur. So, the **yield stress** is a value, while the term **yield strength** indicates a material property.

Machine parts should be designed in such a way that the value of the usual *working stress* σ_{W} has to be always less than the *allowable stress* σ_{AL}. And the allowable stress is determined from

$$\sigma_{\mathrm{AL}} = \sigma_{\mathrm{Y}} / k , \qquad \text{(CR\_61)}$$

where σ_{Y} is the *yield stress* and k is the *factor of safety*.

If the loading stress increases above the σ_{C} value (point C) the material starts to resist again. This part of loading is called the *strain hardening*. The maximum value of stress at point D is called the *ultimate stress*. It is the maximum stress that a material can withstand before it breaks or weakens. During this process the cross-sectional area of the test specimen starts to narrow – this process is called *necking* – and the immediate cross-sectional area, say ${}^{t}S$, is subsequently smaller and smaller being thus different from the original one, i.e. from ${}^{0}S$.

For a still increasing value of the loading force P, the stress in the specimen might be computed by two different ways. The former leads to the definition of so-called *engineering stress*, which is related to the original cross-sectional area by

$$\sigma_{\mathrm{eng}} = \frac{P}{{}^{0}S} . \qquad \text{(CR\_62)}$$

The latter, called the *true stress* (sometimes *Cauchy stress*), is related to the current cross-sectional area, and is defined by

$$\sigma_{\mathrm{true}} = \frac{P}{{}^{t}S}. \qquad \text{(CR\_63)}$$

Both stress representations are depicted in Fig. CR_18. The dotted curve belongs to the true stress, which actually a more 'correct' stress representation. For small strains, however, both stress descriptions are numerically undistinguishable. In the text, we will mostly use the concept of the engineering stress since it is typical for the linear theory of elasticity with infinitesimal strains.

For the further loading beyond the point D (the *ultimate stress*) the failure of the specimen occurs. This is indicated by points E or E' respectively.

4.14. Plasticity

If the loading process is stopped above the yield stress value σ_{Y} – the current value of the strain is just ε as indicated in Fig. CR_19 – and then the loading force is gradually removed, the specimen starts to shorten again. The unloading stress-strain curve goes from the point A to B. Notice that the unloading curve is linear and parallel to the line representing the virgin elastic part of loading. For axial strains, we can write

$$\varepsilon = \varepsilon_{\mathrm{e}} + \varepsilon_{\mathrm{p}}, \qquad \text{(CR\_64)}$$

where the *total strain* ε is composed of two parts – the *elastic (part of) strain* ε_{e} and the *plastic (part of) strain* ε_{p}. The actual shortening of the specimen is associated with the *elastic part of strain* ε_{e} only. If we started the loading process again then the new stress-strain curve would approximately follow the direction from B to A and the stress σ_{A} becomes a new yield stress $\sigma_{\mathrm{y}}^{\mathrm{new}}$. This process is known as the *material hardening*.

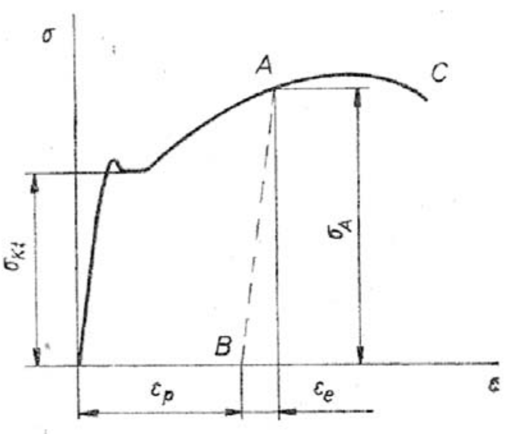


Fig. CR_19 ... Idealized stress strain diagram

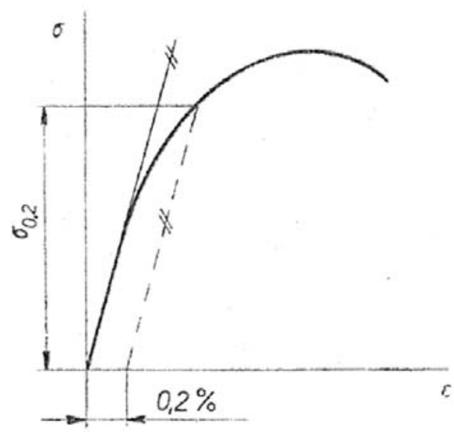
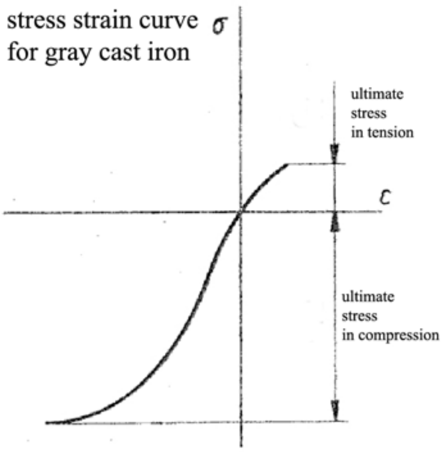


Fig. CR_20 ...Stress strain diagram

Typical values for steel materials are as follows

Steel	Ultimate stress [MPa]	Yield stress
lower strength	350 – 500	about 60 %
medium	700	
high	3000	about 85 %

The stress-strain curve, typical of non-ferrous materials, is shown in Fig. CR_20. In that case, the yield stress point is not clearly pronounced as before and is consensually defined as the stress corresponding to the permanent strain value of 0.2 %.



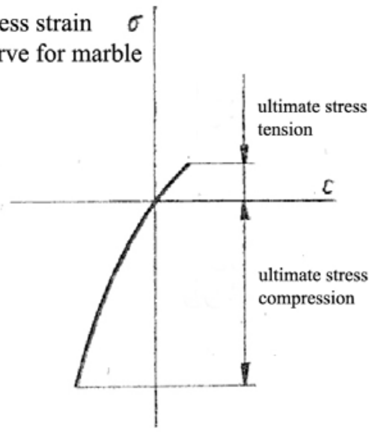


Fig. CR_21 ... Stress strain curve for gray cast iron

Fig. CR_22 ... Stress strain curve for marble

The stress-strain curve for the gray cast iron is in Fig. CR_21. Notice the different material behaviour in tension and compression loadings. Fig. CR_22 shows the stress-strain curve for marble materials.

Of course, the real-time experiments are not as smooth as their idealized appearances shown above.

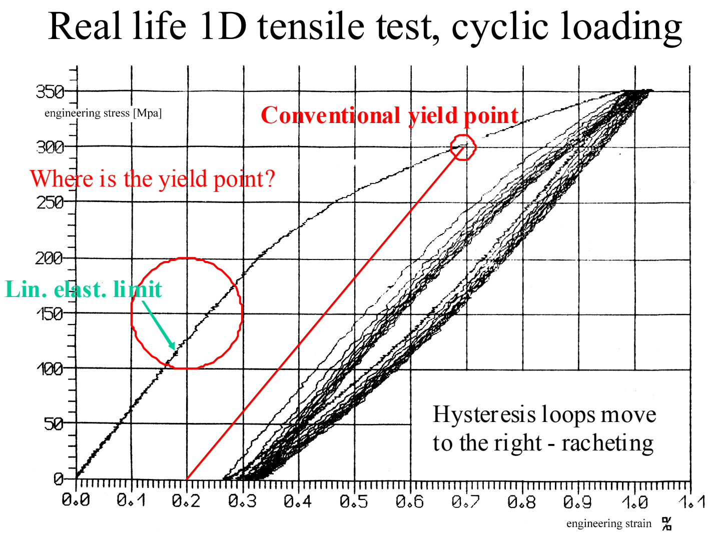


Fig. CR_23 ... Real life tensile test

In Fig. CR_23 there is the record of the actual tensile test for a material not having a pronounced yield stress point and shows how it is determined by applying the above-mentioned rule of 0.2 % strain. Also, it shows what happens when – after a certain above-yield stress is reached – a repeated loading and unloading occur. One can observe that hysteresis loops move step by step to the right – this phenomenon is called the *ratcheting*.

The material properties are substantially influenced by the applied heat treatment of the material. This is shown in Fig. CR_24.

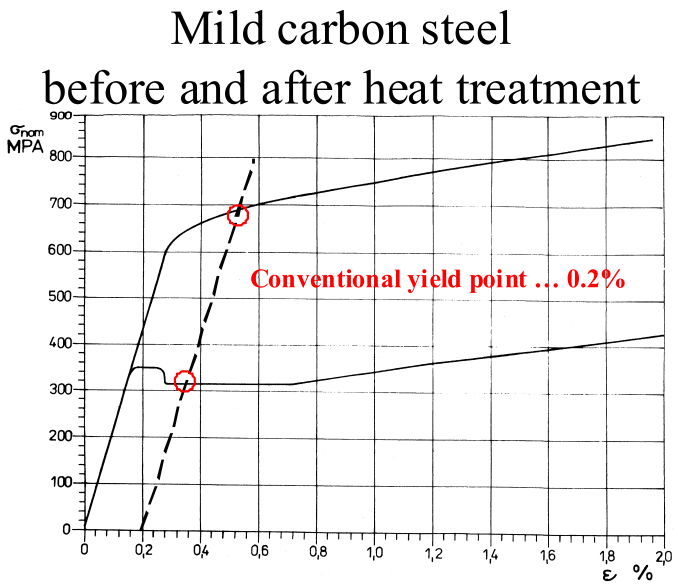


Fig. CR_24 ... Influence of heat treatment on tensile test.

The tensile testing determines the character of uni-axial relations between the axial stress and the axial strain quantities. In 1D these measures are pure scalars. The stress-strain relation in its first part is linear. The coefficient of proportionality between 1D scalar stress and strain quantities is called Young's modulus and is denoted E. It is measured in $[\mathrm{N/m^2}]$.

The testing specimen is not only elongated due to the loading process but it is narrowed as well. So, the pure axial loading produces not only axial deformations but the radial as well.

For more details see [7], [14], [17], [18], [21], [39].

4.15. Axial (longitudinal) vs. radial (transversal, lateral) deformations

Assume that the tested specimen, being clamped in its upper part, has a circular cross-sectional area. In this experiment, the weight of the specimen is neglected.

Due to the applied tensile force F, the specimen is elongated, and at the same time it is contracted – its cross-sectional area is diminished. The overall axial elongation is Δl, while the average radial contraction is Δd. See Fig. CR_25.

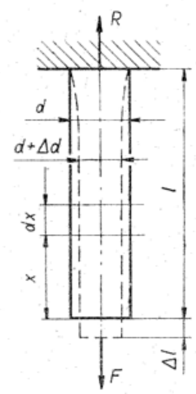


Fig. CR_25 ... Hanging rod

If the rod were compressed instead, its cross-sectional area would increase. The dependence of axial (longitudinal) to radial (transversal) deformations was proved by experiments carried out by Siméon Poisson – 1781–1840. The phenomenon is known by the name of *Poisson's effect*. For small strains, it is quantified by the coefficient μ called the *Poisson's ratio*. It is defined as the ratio of the radial strain to the axial strain. For the radial strain, we can write

$$\varepsilon_{\text{radial}} = \frac{\Delta d}{d} = -\mu\varepsilon_{\text{axial}} = -\mu\frac{\Delta l}{l} = -\mu\frac{\sigma}{E}. \qquad \text{(CR\_65)}$$

This formula is of phenomenological nature. This means that it cannot be proved mathematically. Furthermore, its validity is limited to linear cases.

The minus sign expresses the fact that the positive axial deformation (elongation) is accompanied by the negative radial deformation (narrowing). For most engineering materials, the value of the Poisson's ratio is in the range <0 0.5>. A typical value for steels is about 0.3; for rubber materials, it approaches the value of 0.5. The Poisson's ratio for cork materials is close to zero. These materials show very little radial expansion when compressed – that's why the cork stoppers are advantageously used for corking the wine bottles.

4.16. Hooke's law appearances for different types of loading

4.16.1. Hooke's law for 1D

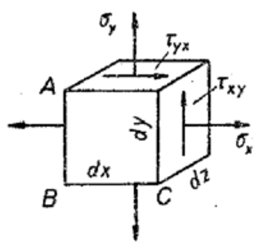
In this case, Hooke's law states that there is a linear relationship between the axial stress component σ_{xx} (sometimes simply σ_x or σ) and the axial strain component ε_{xx} (sometimes simply ε_x or ε). Since there are no other 'alive' directions we often omit the direction indices, because the notation of a particular direction is meaningless, and write the Hooke's law in the scalar form as

$$\sigma = E\varepsilon, \qquad \text{(CR\_66)}$$

where the coefficient of proportionality E is called the Young's modulus.

4.16.2. Hooke's law for the plane stress

The considered stress components, acting on the elementary cube, are depicted in Fig.CR_26.



The corresponding strain component in the x direction consists of the elongation due to the σ_x stress component accompanied by the shortening due to the σ_y stress component.

$$\varepsilon_x = \frac{\Delta \mathrm{d}x}{\mathrm{d}x} = \frac{1}{E}\left(\sigma_x - \mu\sigma_y\right). \qquad \text{(CR\_67)}$$

Fig. CR_26 ... Plane stress

Analogically, for the y - direction

$$\varepsilon_y = \frac{\Delta \mathrm{d}y}{\mathrm{d}y} = \frac{1}{E}\left(\sigma_y - \mu\sigma_x\right). \qquad \text{(CR\_68)}$$

Even if we assume that there is no stress component in the z-direction, the considered element – due to applied positive stresses σ_x, σ_y has to shorten in the z direction. This phenomenon was experientially proved by the French mathematician and engineer Siméon Denis Poisson (1781 – 1840). Thus, the strain component in the z - direction is

$$\varepsilon_z = -\frac{\mu}{E}\left(\sigma_x + \sigma_y\right). \qquad \text{(CR\_69)}$$

We have shown that the change of the right angle of the considered element, expressed by the shear strain, is proportional to the shear strain.

$$\gamma_{xy} = \gamma_{yx} = \frac{1}{G}\tau_{xy} = \frac{1}{G}\tau_{yx}. \qquad \text{(CR\_70)}$$

So far, we expressed the strain components as functions of the stress components. The inverse relations can be easily derived and have the form

$$\sigma_x = \frac{E}{1-\mu^2}\left(\varepsilon_x + \mu\varepsilon_y\right), \qquad \text{(CR\_71a)}$$

$$\sigma_y = \frac{E}{1-\mu^2}\left(\varepsilon_y + \mu\varepsilon_x\right), \qquad \text{(CR\_71b)}$$

$$\tau_{xy} = \tau_{yx} = G\gamma_{xy} = G\gamma_{yx} = \frac{E}{2(1+\mu)}\gamma_{xy}. \qquad \text{(CR\_71c)}$$

Hooke's law for the plane stress in the matrix form is

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \dfrac{1-\mu}{2} \end{bmatrix}\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}. \qquad \text{(CR\_72)}$$

4.16.3. Hooke's law for 3D

3D state of stress

For a body being loaded by a 3D system of forces we generally have nine stress components. Their action on an elementary cube is depicted in Fig. CR_27. The triple trio of stress components is called the tensor. This is how the stress tensor components might be denoted and expressed in various matrix forms

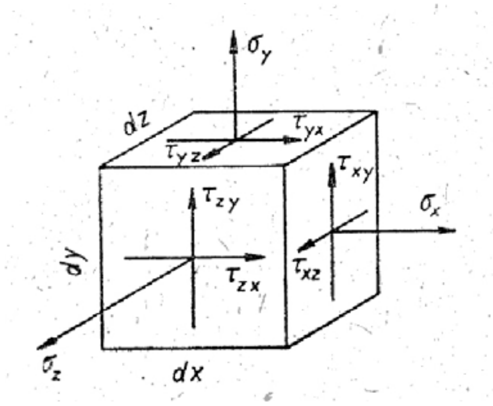


Fig. CR_27 ... Stress components in 3D.

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yz} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yz} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \boldsymbol{\sigma}. \qquad \text{(CR\_73)}$$

The strain components in 3D – engineering notation

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \mu\sigma_y - \mu\sigma_z\right),$$
$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \mu\sigma_z - \mu\sigma_x\right),$$
$$\varepsilon_z = \frac{1}{E}\left(\sigma_z - \mu\sigma_x - \mu\sigma_y\right),$$
$$\gamma_{xy} = \frac{1}{G}\tau_{xy} = \frac{2(1+\mu)}{E}\tau_{xy},$$
$$\gamma_{yz} = \frac{1}{G}\tau_{yz} = \frac{2(1+\mu)}{E}\tau_{yz},$$
$$\gamma_{zx} = \frac{1}{G}\tau_{zx} = \frac{2(1+\mu)}{E}\tau_{zx}. \qquad \text{(CR\_74a … CR74f)}$$

Due to symmetry

$$\varepsilon_{xx} = \varepsilon_x, \varepsilon_{yy} = \varepsilon_y, \varepsilon_{zz} = \varepsilon_z, \varepsilon_{xy} = \gamma_{xy}, \varepsilon_{yz} = \gamma_{yz}, \varepsilon_{zx} = \gamma_{zx}. \qquad \text{(CR\_75)}$$

The strain components in 3D – matrix notation

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\mu & -\mu & 0 & 0 & 0 \\ -\mu & 1 & -\mu & 0 & 0 & 0 \\ -\mu & -\mu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\mu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\mu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\mu) \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix}; \ \boldsymbol{\varepsilon} = \mathbf{D}\boldsymbol{\sigma}$$

… (CR_76)

The strain components for plane stress – see Fig. CR_26.

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\mu & 0 \\ -\mu & 1 & 0 \\ 0 & 0 & 2(1+\mu) \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}; \ \boldsymbol{\varepsilon} = \mathbf{D}\boldsymbol{\sigma}\,. \qquad \text{(CR\_77)}$$

The element is contracted in z - direction, so $\varepsilon_{33} = -\frac{\mu}{E}\left(\sigma_{11} + \sigma_{22}\right)$. (CR_78)

Summary for Hooke's law representations

Hooke's law in 3D space

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \mathbf{E} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{Bmatrix} \qquad \mathbf{E} = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 & 0 & 0 \\ \mu & 1-\mu & \mu & 0 & 0 & 0 \\ \mu & \mu & 1-\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\mu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix}$$

… (CR_79)

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\mu & -\mu & 0 & 0 & 0 \\ -\mu & 1 & -\mu & 0 & 0 & 0 \\ -\mu & -\mu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\mu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\mu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\mu) \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix}. \qquad \text{(CR\_80)}$$

4.16.4. Plane stress vs. plane strain

Depending on how the elementary cube is constrained we distinguish two cases. See Fig. CR_28.

If the face ABC is free, then we are dealing with so-called *plane stress state of stress*. If the face ABC, and its parallel face, are fixed (no displacements allowed) then we have the case called *plane strain state of stress*. In the case of the plane stress the material element is allowed to freely deform in the z-direction and thus the displacement u_z and the strain ε_{zz} are non-zero, while the corresponding stress σ_{zz} is equal to zero. In the plane strain case, the material element is constrained in its *z*-direction motion and thus $\varepsilon_{zz} = 0$ and consequently $\sigma_{zz} \neq 0$.

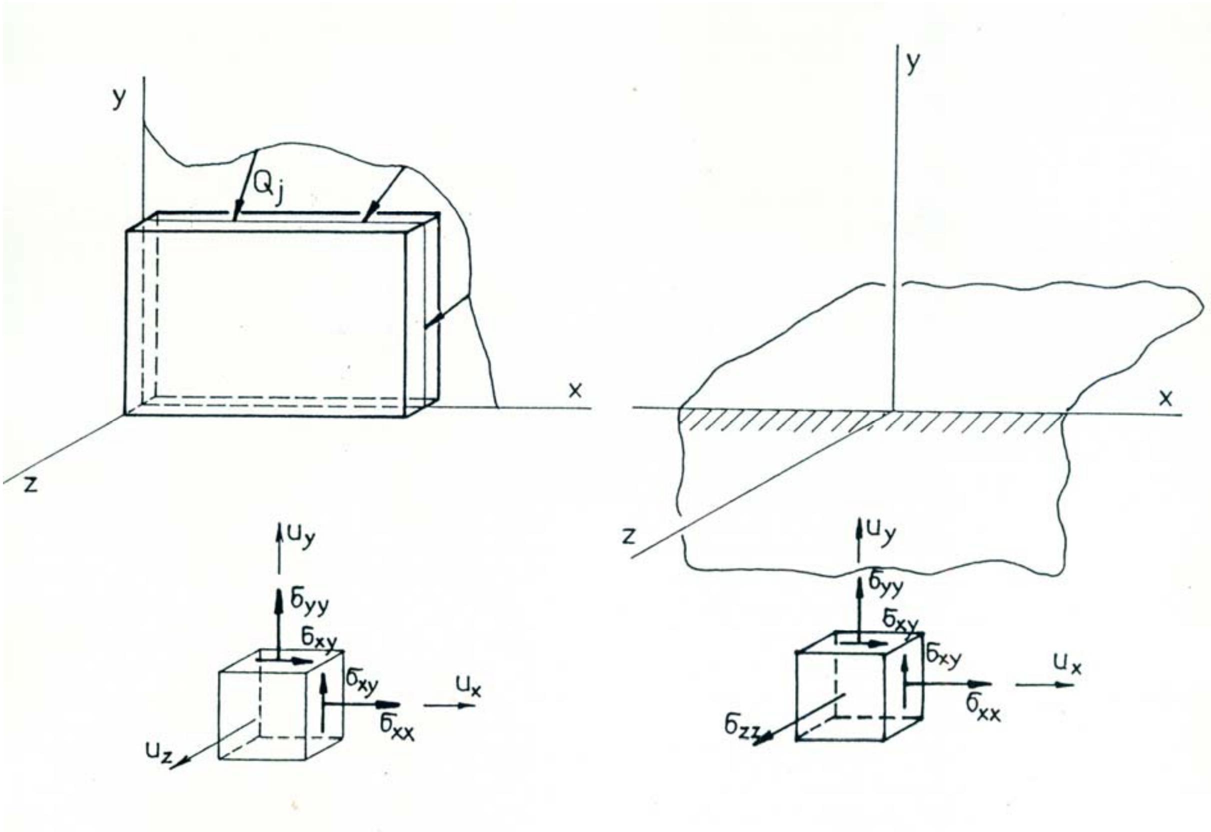


Fig. CR_28_ ... Plane stress and strain

Hooke's law for plane stress ($\sigma_{33} = \sigma_{23} = \sigma_{31} = 0$, $\varepsilon_{23} = \varepsilon_{31} = 0$)

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \dfrac{1-\mu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix}. \tag{CR\_81}$$

The element changes its dimensions in z - direction, so

$$\varepsilon_{33} = -\frac{\mu}{E}(\sigma_{11} + \sigma_{22}) = -\frac{\mu}{1-\mu}(\varepsilon_{11} + \varepsilon_{22}). \tag{CR\_82}$$

Hooke's law for plane strain $(\sigma_{23} = \sigma_{31} = 0; \quad \varepsilon_{33} = \varepsilon_{31} = \varepsilon_{23} = 0)$.

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & 0 \\ \mu & 1-\mu & 0 \\ 0 & 0 & \dfrac{1-2\mu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix} \qquad \text{(CR\_83)}$$

and also $\sigma_{33} = \mu(\sigma_{11} + \sigma_{22})$. (CR_84)

4.17. Principal stresses – once more

The stress vector[5] $\mathbf{f}$ acting in the plane ABC, as depicted in Fig. CR_29, can be decomposed into a component in the direction of the normal $\mathbf{n}$ i.e. $\sigma \mathrm{d}S$, and into another component, i.e. $\tau \mathrm{d}S$, lying in the plane ABC. The normal $\mathbf{n}$ is defined by its direction cosines, i.e.

$$\begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} \cos\alpha_1 \\ \cos\alpha_2 \\ \cos\alpha_3 \end{Bmatrix}. \qquad \text{(CR\_84)}$$

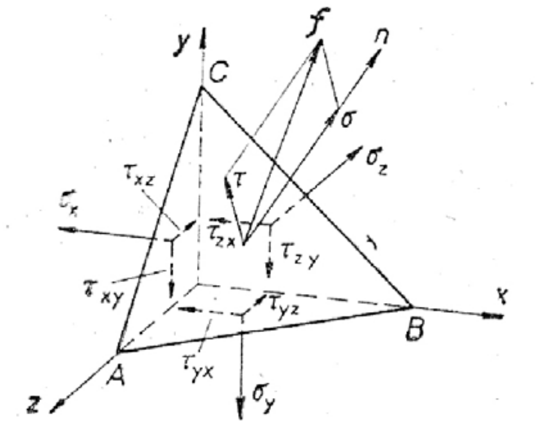


Fig. CR_29 ... Stress vectors

The stress tensor could be expressed in the matrix form[6] as

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}. \qquad \text{(CR\_85)}$$

The stress vector components f_i are related to components of the stress tensor σ_{ij} by

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}; \; \mathbf{f} = \boldsymbol{\sigma}\,\mathbf{n}; \; f_i = \sigma_{ij} n_j. \qquad \text{(CR\_86)}$$

This expression, called the Cauchy relation, is based on satisfying the force equilibrium conditions in directions of coordinate axes.

Now, we are looking for such a position of ABC plane, in which the stress vector $\mathbf{f}$ would be perpendicular to that plane. In other words, the stress vector would have the same direction as

[5] Stress is a tensor. Stress vector is a rarely used in engineering computations, but it is a useful entity allowing to express the relations between forces and stress components. The stress vector dimension is in $[\mathrm{N/m^2}]$.

[6] The indices *x,y,z* are replaced by 1,2,3.

the normal vector $\mathbf{n}$ and its tangential (i.e. shear) components disappear. Under these conditions, the stress vector $\mathbf{f}$ would become a scalar multiple of the normal $\mathbf{n}$.

So we require that

$\mathbf{f} = \lambda \mathbf{n}$ or expressed in components $f_i = \lambda n_i$, (CR_87)

where the so far unknown scalar multiple is denoted λ.

Beforehand, we could claim that the disappearance of shear stress components would lead to the diagonal form of the original tensor $\boldsymbol{\sigma}$.

Substituting Eq. (CR_87) to Eq. (CR_86) we get

$\lambda \mathbf{n} = \boldsymbol{\sigma} \mathbf{n}$ and consequently $\lambda = \mathbf{n}^{\mathrm{T}} \boldsymbol{\sigma} \mathbf{n}$. (CR_88)

This is, however, the way how the eigenvalue problem is defined in mathematics. The geometrical meaning is: find such an eigenvalue λ and such a normal vector $\mathbf{n}$, called eigenvector, which causes that the Eq. (CR_88) is satisfied. Generally, there are n eigenvalues and n eigenvectors for a matrix of the order $n \times n$.

Using the unit matrix, Eq. (CR_88) could be rewritten into the form

$\boldsymbol{\sigma}\, \mathbf{n} - \boldsymbol{\lambda} \mathbf{I} \mathbf{n} = \mathbf{0}$ or $(\boldsymbol{\sigma} - \lambda \mathbf{I})\mathbf{n} = \mathbf{0}$. (CR_89a)

This is a system of homogeneous algebraic equations which has a non-trivial solution only if the determinant of the system is identically equal to zero. Thus

$$\left| \boldsymbol{\sigma} - \lambda \mathbf{I} \right| = \begin{vmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \lambda \end{vmatrix} = 0. \qquad \text{(CR\_89b)}$$

We have already stated that the matrix is symmetric, so $\sigma_{ij} = \sigma_{ji}$.

Evaluating Eq. (CR_89b) leads to the cubic equation

$\lambda^3 - J_1 \lambda^2 + J_2 \lambda - J_3 = 0$, (CR_90)

Solving Eq. (CR_90) leads to three real roots $\lambda_1, \lambda_2, \lambda_3$ – these roots are called the eigenvalues of $\boldsymbol{\sigma}$. The constants J_i, known as the stress invariants, have the form

$J_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$, (CR_91a)

$$J_2 = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{32} & \sigma_{33} \end{vmatrix}, \qquad \text{(CR\_91b)}$$

$$J_3 = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix}. \qquad \text{(CR\_91c)}$$

For each eigenvalue, we can write

$$\lambda_i = (\mathbf{n}^{(i)})^{\mathrm{T}} \sigma \ \mathbf{n}^{(i)}, \ i = 1,2,3. \qquad \text{(CR\_92)}$$

Alternatively,

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} = \begin{bmatrix} n_1^{(1)} & n_2^{(1)} & n_3^{(1)} \\ n_1^{(2)} & n_2^{(2)} & n_3^{(2)} \\ n_1^{(3)} & n_2^{(3)} & n_3^{3} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1^{(1)} & n_1^{(2)} & n_1^{(3)} \\ n_2^{(1)} & n_2^{(2)} & n_2^{(3)} \\ n_3^{(1)} & n_3^{(2)} & n_3^{(3)} \end{bmatrix}. \qquad \text{(CR\_93)}$$

Or, in a more compact form

$$\lambda = \mathbf{N}^{\mathrm{T}} \sigma \ \mathbf{N}, \text{ where } \mathbf{N} = \left[\mathbf{n}^{(1)} \ \mathbf{n}^{(2)} \ \mathbf{n}^{(3)}\right] \text{ and } \mathbf{n}^{(i)} = \begin{Bmatrix} \cos\alpha_1^{(i)} \\ \cos\alpha_2^{(i)} \\ \cos\alpha_3^{(i)} \end{Bmatrix}. \qquad \text{(CR\_94)}$$

The eigenvalues λ_i of the matrix $\boldsymbol{\sigma}$ correspond to the principal stress components σ_i. The matrix $\mathbf{N}$ is known as the eigenvector matrix. In this case, it has three columns containing the vectors $\mathbf{n}^{(i)}$. The components of these vectors contain the cosines of direction angles determining the vector orientation with respect to coordinate axes. These angles (we have three triples of them) determine the cross section orientations in which the shear stresses disappear – the $\boldsymbol{\sigma}$ matrix becomes diagonal and their diagonal entries (eigenvalues) are the principal stresses.

It was shown how the principal stresses and their orientations can be determined by means of the Mohr's circle reasoning. The process could be substantially simplified by numerical techniques available in Matlab. To find the eigenvectors and eigenvalues of a matrix, say $\boldsymbol{\sigma}$, it suffices to write the command

```
[N, Lambda] = eig(Sigma);
```

It is understood that `N` … $\mathbf{N}$, `Lambda` … $\boldsymbol{\lambda}$ and `Sigma` … $\boldsymbol{\sigma}$.

05_SE. Strain energy

5.1 Introduction

In this paragraph, we are dealing with infinitesimal strains and engineering stresses. Also, the validity of the Hooke's law is assumed.

In continuum mechanics, the *strain energy* is the internal energy accumulated in a body being deformed.

5.2. Strain energy for uniaxial stress

An elementary cube being loaded in the *x*-direction only by a force P_x is depicted in Fig. SE_1.

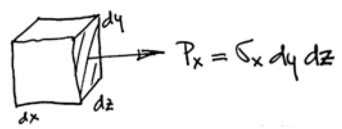


Fig. SE_1 … 1D elem cube loaded in x

Due to the applied loading, the side of the cube $\mathrm{d}x$ is elongated by $\Delta \mathrm{d}x$. So, the corresponding strain in this direction is $\varepsilon_x = \dfrac{\Delta \mathrm{d}x}{\mathrm{d}x}$. The acting force could be expressed by means of the stress σ_x and the corresponding cross-sectional area $S = \mathrm{d}y\,\mathrm{d}z$, and the elementary force $P_x = \sigma_x \,\mathrm{d}y\,\mathrm{d}z$. One-dimensional appearance of the Hooke's law is $\sigma_x = E\varepsilon_x$ and could, in this case, be reformulated as $P_x = \dfrac{ES}{\mathrm{d}x}\Delta \mathrm{d}x$. This formulation states that the elongation of the cube side $\Delta \mathrm{d}x$ is proportional to the loading force P_x. The coefficient of proportionality, say $k = \dfrac{ES}{\mathrm{d}x}$, is called the *stiffness*.

During to the loading process, the force P_x linearly increases from zero to its maximum value $P_{\max}$, while the corresponding elongation u increases from zero to the maximum value $\Delta \mathrm{d}x$. Knowing the stiffness, we could plot the *force-displacement* line, i.e. $P_x = ku$, as seen in Fig. SE_2.

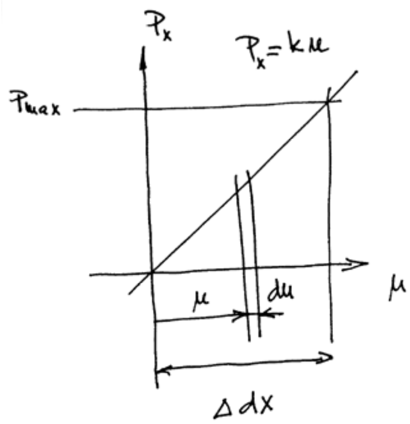


Fig. SE_2 … 1D force displacement line.

It is known that the mechanical work, which is an equivalent of the mechanical energy, can be expressed as a scalar product of the force and the displacement. Since the loading force varies, the elementary work $\mathrm{d}U$ done by the force P_x during the elongation of the side of the cube by $\Delta \mathrm{d}x$, has to be evaluated in the integral fashion, as

$$\mathrm{d}U=\int_0^{\Delta\mathrm{d}x}P_x\,\mathrm{d}u=\int_0^{\Delta\mathrm{d}x}ku\,\mathrm{d}u=\frac{1}{2}k\left[u^2\right]_0^{\Delta\mathrm{d}x}=\frac{1}{2}\frac{ES}{\mathrm{d}x}(\Delta\mathrm{d}x)^2=\frac{1}{2}\frac{E\,\mathrm{d}y\,\mathrm{d}z}{\mathrm{d}x}\varepsilon_x^2(\mathrm{d}x)^2= \qquad \text{... (SE\_1)}$$
$$=\frac{1}{2}E\varepsilon_x^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\frac{1}{2}E\varepsilon_x^2\,\mathrm{d}V.$$

Realizing that $\sigma_x = E\varepsilon_x$ the relation for the *elementary strain energy* could be further elaborated as follows

$$\mathrm{d}U=\frac{1}{2}E\varepsilon_x^2\,\mathrm{d}V=\frac{1}{2}E\frac{\sigma_x^2}{E^2}\mathrm{d}V=\frac{1}{2}\frac{\sigma_x^2}{E}\mathrm{d}V=\frac{1}{2}\frac{\sigma_x}{E}E\varepsilon_x\,\mathrm{d}V=\frac{1}{2}\sigma_x\varepsilon_x\,\mathrm{d}V\,. \qquad \text{(SE\_2)}$$

The *strain energy density* is the strain energy that is related to a unit of volume, thus

$$\Lambda=\frac{\mathrm{d}U}{\mathrm{d}V}=\frac{1}{2}E\varepsilon_x^2=\frac{1}{2}E\frac{\sigma_x^2}{E^2}=\frac{1}{2}\frac{\sigma_x^2}{E}=\frac{1}{2}\frac{\sigma_x}{E}E\varepsilon_x=\frac{1}{2}\sigma_x\varepsilon_x\ . \qquad \text{(SE\_3)}$$

5.3. Strain energy for plane state of stress

An elementary cube in the state of the plane stress is depicted in Fig. SE_3.

We have already shown that this type of loading evokes the normal strains, i.e. $\varepsilon_x, \varepsilon_y$, and the shear strains, i.e γ_{xy}.

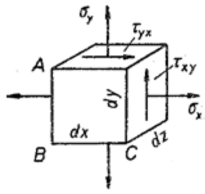


Fig. SE_3 … Plane stress

In the linear theory of elasticity, the strain energy due normal strains and the strain energy due to the shear strains are independent and could be superimposed.

5.3.1. The strain energy due to normal stresses

The total elongation $\Delta\mathrm{d}x$ of the cube side $\mathrm{d}x$ is composed of the elongation due the stress σ_x and of the shortening due the perpendicularly acting stress σ_y. Thus,

$$\Delta\mathrm{d}x=\varepsilon_x\,\mathrm{d}x=\frac{1}{E}\left(\sigma_x-\mu\sigma_y\right)\mathrm{d}x\,. \qquad \text{(SE\_4)}$$

Similarly, for the y-direction

$$\Delta\mathrm{d}y=\varepsilon_y\,\mathrm{d}y=\frac{1}{E}\left(\sigma_y-\mu\sigma_x\right)\mathrm{d}y\,. \qquad \text{(SE\_5)}$$

The strain energies due to the normal stresses are composed of two parts.

- The work done by the force $\sigma_x\mathrm{d}y\,\mathrm{d}z$ due to the elongation $\Delta\mathrm{d}x$ is $\sigma_x\varepsilon_x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.
- The work done by the force $\sigma_y\mathrm{d}z\,\mathrm{d}x$ due to the elongation $\Delta\mathrm{d}y$ is $\sigma_y\varepsilon_y\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

Since the stress components and the corresponding strain components are mutually perpendicular their scalar products do not influence each other.

Generally, the loading process occurs in time – it is assumed that both stress and the strain quantities rise linearly from zero to their maximum values[1]. Let's define an auxiliary parameter, say $\lambda = \lambda(t)$, $0 \le \lambda \le 1$, which formally describes the loading and deformation processes in time. Then, the increments of exerted work (strain energy) for immediate values of the forces $\lambda\sigma_x \mathrm{d}y\,\mathrm{d}z$ and $\lambda\sigma_y \mathrm{d}z\,\mathrm{d}x$ by incremental elongations $\mathrm{d}\lambda\,\varepsilon_x\,\mathrm{d}x$ and $\mathrm{d}\lambda\varepsilon_y\,\mathrm{d}y$ respectively, are

$$(\lambda\sigma_x\ \mathrm{d}y\ \mathrm{d}z)\ (\mathrm{d}\lambda\varepsilon_x\ \mathrm{d}x\ ) = \sigma_x\varepsilon_x\ \lambda\,\mathrm{d}\lambda\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \sigma_x\varepsilon_x\ \lambda\mathrm{d}\lambda\ \mathrm{d}V\,, \qquad \text{(SE\_6a)}$$

$$(\lambda\sigma_y\ \mathrm{d}x\ \mathrm{d}z)\ (\mathrm{d}\lambda\varepsilon_y\ \mathrm{d}y\ ) = \sigma_y\varepsilon_y\ \lambda\mathrm{d}\lambda\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \sigma_y\varepsilon_y\ \lambda\mathrm{d}\lambda\ \mathrm{d}V\ . \qquad \text{(SE\_6b)}$$

The *elementary strain energy* due to the normal stresses is a cumulative process that can be evaluated by the integration process in "time"

$$\mathrm{d}A_\sigma = \left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y\right) dV \int\limits_{\lambda=0}^{1} \lambda\ \mathrm{d}\lambda = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y\right) dV\ . \qquad \text{(SE\_7)}$$

Often, we define the *strain energy density*, which is related to a unit of volume

$$\Lambda_\sigma = \frac{\mathrm{d}A_\sigma}{\mathrm{d}V} = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y\right) \ \text{ or }\ \Lambda_\sigma = \frac{1}{2E}\left(\sigma_x^2 + \sigma_y^2 - 2\mu\sigma_x\sigma_y\right). \qquad \text{(SE\_8)}$$

The formula on the right was obtained by substituting the Hooke's law relations, i.e.

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \mu\sigma_y\right) \ \text{and}\ \varepsilon_y = \frac{1}{E}\left(\sigma_y - \mu\sigma_x\right)\ . \qquad \text{(SE\_9)}$$

5.3.2. The strain energy due to shear stress

A part of the strain energy attributed to the shear strains can be deduced analogically in the form

$$\mathrm{d}A_\tau = \frac{1}{2}\tau_{xy}\gamma_{xy}\ dV\ . \qquad \text{(SE\_10)}$$

Similarly, the shear strain energy density is

$$\Lambda_\tau = \frac{\mathrm{d}A_\tau}{\mathrm{d}V} = \frac{1}{2}\tau_{xy}\gamma_{xy} = \frac{\tau_{xy}^2}{2G}\,. \qquad \text{(SE\_11)}$$

The total shear strain energy density for the plane stress is

[1] In statics, we normally do not take the time variable into consideration. The applied force is either zero or it has its maximum value. The process of the force application is, however, assumed to be so slow that the inertia effects could be neglected.

$$\Lambda = \Lambda_{\sigma} + \Lambda_{\tau} = \frac{1}{2}\left(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy}\gamma_{xy}\right). \qquad \text{(SE\_12)}$$

5.4. Strain energy for the 3D state of stress

Analogically, the strain energy density for the 3D state of stress, depicted in Fig. SE_4, is

$$\Lambda = \frac{1}{2}\left(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z\right) + \frac{1}{2}\left(\tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right). \qquad \text{(SE\_13)}$$

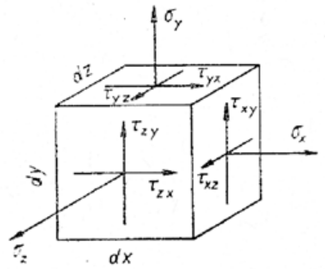


Fig. SE_4 … 3D stress

Eliminating strains, using the Hooke's law relations, we get

$$\Lambda = \frac{1}{2E}\left[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\mu\left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x\right)\right] + \frac{1}{2G}\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right). \qquad \text{(SE\_14)}$$

5.5. Strain energy in a beam subjected to pure bending

In Fig. SE_5 there is depicted a part of the beam subjected to pure bending. We have derived that the axial stress due to the bending is

$$\sigma_x = \frac{M(x)}{J_y} z. \qquad \text{(SE\_15)}$$

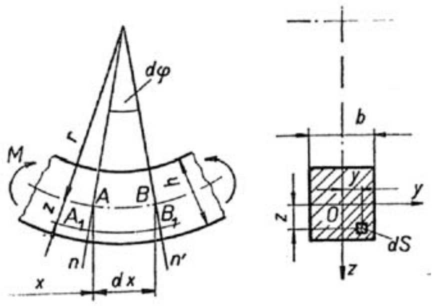


Fig. SE_5 … Beam defo 1

The pure bending means that the axial stress is of uniaxial nature, and that the influence of shear forces is non-existent or neglected. So, the strain energy of pure bending is analogous to the strain energy in tension – compression as explained before. See Fig. SE_6. So, the elementary strain energy, contained in an element of a beam between two infinitesimally close slices, depicted in Fig. SE_5, is

$$\mathrm{d}U = \left(\frac{\sigma_x^2}{2E}\mathrm{d}S\right)\mathrm{d}x = \left[\left(\frac{M^2(x)}{2EJ_y^2}\right)\int_S z^2\,\mathrm{d}S\right]\mathrm{d}x =$$

$$= \frac{M^2(x)}{2EJ_y}\mathrm{d}x. \qquad \text{(SE\_16)}$$

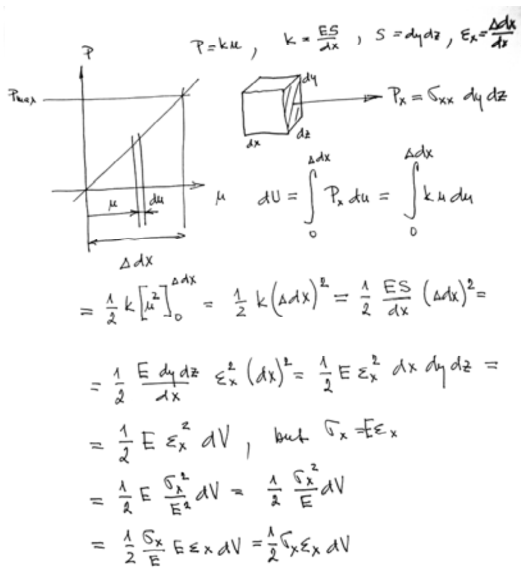


Fig. SE_6 … 1D strain energy

To explain the analogy of bending with the tension it should be reminded that for a bar of the length l we write

$$\sigma = E\varepsilon; \quad \frac{P}{S} = E\frac{\Delta l}{l}; \quad P = \frac{ES}{l}\Delta l . \tag{SE\_17}$$

And similarly for a beam element of the length $\mathrm{d}x$

$$P = \frac{ES}{\mathrm{d}x}\Delta \mathrm{d}x; \quad P = k\,\Delta \mathrm{d}x; \quad k = \frac{ES}{\mathrm{d}x}. \tag{SE\_18}$$

The strain energy contained in the whole beam is obtained by the integrating Eq. (SE_16). If $J_y = const$, then

$$U = \frac{1}{2EJ_y}\int_0^l M^2(x)\,\mathrm{d}x . \tag{SE\_19}$$

5.6. Strain energy expressed in tensor notation

It is of interest that all the strain energy density formulations for the cases examined in this paragraph could be simply and uniquely described by the tensor analysis notation. In the form of the tensor scalar product, also known as the tensor double product, we have

$$\Lambda = \frac{1}{2}\Sigma_{ij}^{\text{engineering}}\, E_{ij}^{\text{Cauchy}} \quad \text{or} \quad \Lambda = \frac{1}{2}\mathbf{\Sigma}^{\text{engineering}} : \mathbf{E}^{\text{Cauchy}} .$$

Here, it is necessary to use one’s wits and to properly distinguish the tensor and Voigt’s (engineering) notations when the tensor and engineering formulas for the strain energy are alternatively employed for the strain energy computation. Of course, the result has to be same.

The Green-Lagrange strain tensor, multiplied by the second Piola-Kirchhoff stress tensor – by means of the double dot product, i.e. $\frac{1}{2}\,{}_0^tE_{ij}\;{}_0^tS_{ij}$ – gives the scalar quantity which represents the mechanical energy or the mechanical work.

5.7. Analogy of relations for tension, bending and torsion

1D stress	bending	torsion
$\sigma = E\varepsilon$	$\sigma = E\varepsilon$	$\tau = G\gamma$ ... Hooke's law
$\sigma = \dfrac{F}{S} = E\varepsilon = E\dfrac{\Delta l}{l}$	$\sigma = \dfrac{M_{\mathrm{o}}}{W_{\mathrm{o}}}$	$\tau = \dfrac{M_{\mathrm{k}}}{W_{\mathrm{k}}}$ … stress

where M_{o} and M_{k} are bending and torsion moments respectively

S	$W_{\mathrm{o}} = \dfrac{J_y}{z_{\mathrm{max}}}$	$W_{\mathrm{k}} = \dfrac{J_{\mathrm{p}}}{r_{\mathrm{max}}}$
area	these relations are valid for circular cross sections only W_{o} and W_{k} are section modules in bending and torsion	

longitudinal strain	curvature	rate of twist
$\varepsilon = \dfrac{\Delta l}{l} = \dfrac{F}{ES}$	$\dfrac{1}{\rho} = \dfrac{M_{\mathrm{o}}}{EJ_y}$	$\vartheta = \dfrac{\mathrm{d}\varphi}{\mathrm{d}x} = \dfrac{M_{\mathrm{k}}}{GJ_{\mathrm{p}}}$

stiffness

$F = \underbrace{\dfrac{ES}{l}}_{\text{longitud. stiffness}} \underbrace{\Delta l}_{\text{elongation}}$	$M_{\mathrm{o}} = \underbrace{EJ_y}_{\text{bending stiffness}} \underbrace{\dfrac{1}{\rho}}_{\text{curvature}}$	$M_{\mathrm{k}} = \underbrace{\dfrac{GJ_{\mathrm{p}}}{l}}_{\text{torsional stiffness}} \underbrace{\varphi}_{\text{twist}}$

strength theories

$\sigma_{\mathrm{max}} = E\varepsilon_{\mathrm{max}} < \sigma_{\mathrm{Dt}}$	$\sigma_{\mathrm{max}} = \dfrac{M_{\mathrm{o}}}{W_{\mathrm{o}}} < \sigma_{\mathrm{Dt}}$	$\tau_{\mathrm{max}} = \dfrac{M_{\mathrm{k}}}{W_{\mathrm{k}}} < \tau_{\mathrm{D}}$

where σ_{Dt} is the allowable stress in tension and τ_{D} is the allowable stress in torsion.

strain energy

$U = \dfrac{F^2 l}{2ES}$	$U = \displaystyle\int_0^l \dfrac{M_o^2(x)}{2EJ_y}\mathrm{d}x$	$U = \dfrac{M_{\mathrm{k}}^2 l}{2GJ_{\mathrm{p}}}$
constant force	variable moment	constant moment

For more details see [17], [18], [19], [22], [23], [33], [34], [36].

06_FT. Failure theories

6.0. Introduction

It should be reminded again that the term the strength of material is understood in two distinct meanings. First, the subject of the university course dedicated to the engineering continuum solid mechanics, also known under the name the mechanics of material. Second, the property of a particular material to withstand safely the applied loading.

Having learnt how

- the loading modes are classified,
- a body is strained due to the applied loading,
- to compute the stress and strain invariants,
- to assess the strain energies for particular loading modes,
- to evaluate the principal strains and stresses,
- the fundamental material constants are experimentally obtained,

we are ready to analyze the conditions under which the examined body could safely withstand the applied loading.

The most common material test is the tensile test, which was briefly described in Paragraph 04_CR. This type of test provides the material constants for 1D loading. The question arises how to apply the 1D material data to cases where the 2D and 3D state of stress occur.

The problem is rather complicated since at each material point (particle) of a loaded body the state of stress is described by a stress tensor, the quantity generally having nine stress and nine strain components. How to decide which stress component is crucial for the capability of the body to withstand the applied loading? That's why the scalar quantities, as the principal stresses, stress invariants, etc. are important. Based on these scalar quantities, the various failure theories and hypotheses were derived. In the following text, a brief survey is presented.

6.1. Rankine's hypothesis of the maximum stress

The spatial state of stress is compared with the uniaxial one in such a way that the maximum stresses are compared. If the principal stresses are ordered as $\sigma_1 \ge \sigma_2 \ge \sigma_3$, then the failure occurs if $\sigma_1 = \sigma_{\mathrm{Pt}}$, where σ_{Pt} is the stress corresponding to the allowable tensile strength. If $\sigma_3 < 0$, then $|\sigma_3| = \sigma_{\mathrm{Pd}}$, where σ_{Pd} is the stress corresponding to the allowable compressive strength. So,

Rankine	(FT_1)
If $\sigma_1 \ge \sigma_2 \ge \sigma_3 \ge 0$, then the failure occurs for $\sigma_1 = \sigma_{\mathrm{Pt}}$.	
If $\sigma_1 \ge \sigma_2 \ge \sigma_3$ and at the same time $\sigma_3 < 0$, then the failure occurs for $|\sigma_3| = \sigma_{\mathrm{Pd}}$.	

6.2. Saint-Venant's hypothesis for the maximum shear

This hypothesis assumes that the failure occurs if the maximum strain reaches the critical value $\varepsilon_{\text{crit}}$. Again, the uniaxial and the spatial state of stress are compared. Assume that the critical stress σ_0 evokes the critical strain $\varepsilon_{\max} = \dfrac{\sigma_0}{E}$. Similarly, if the principal stresses are ordered as $\sigma_1 \geq \sigma_2 \geq \sigma_3$ then the maximum strain will be $\varepsilon_{\max} = \dfrac{1}{E}(\sigma_1 - \mu\sigma_2 - \mu\sigma_3)$. Comparing them, we get $\dfrac{\sigma_0}{E} = \dfrac{1}{E}(\sigma_1 - \mu\sigma_2 - \mu\sigma_3)$. From it follows

$$\sigma_0 = (\sigma_1 - \mu\sigma_2 - \mu\sigma_3) = \sigma_{\text{Pt}}, \qquad \text{(FT\_2)}$$

where σ_{Pt} is the allowable tensile strength. The value σ_0 is often called the equivalent stress.

Saint-Venant (FT_3)
If $\sigma_1 \geq \sigma_2 \geq \sigma_3$, then the failure occurs for $\sigma_0 = \sigma_{\text{Pt}}$, where $\sigma_0 = (\sigma_1 - \mu\sigma_2 - \mu\sigma_3)$.

6.3. Guest's hypothesis of the maximum shear stress

For the uni-axial state of stress we get $\tau_{\max} = \sigma_0 / 2$. For the spatial state of stress, assuming that the principal stresses are ordered as $\sigma_1 \geq \sigma_2 \geq \sigma_3$, the maximum shear stress is $\tau_{\max} = (\sigma_1 - \sigma_3)/2$. Comparing we get that the equivalent stress is $\sigma_0 = \sigma_1 - \sigma_3$.

Guest (FT_4)
If $\sigma_1 \geq \sigma_2 \geq \sigma_3$, then the failure occurs if $\sigma_0 = \sigma_{\text{Pt}}$,
where $\sigma_0 = \sigma_1 - \sigma_3$ and σ_{Pt} is the allowable tensile strength.

6.4. Beltrami's hypothesis of the strain energy density

According to this hypothesis, two states of stress are equivalent, if their strain energy densities are equal. For the uniaxial and for the spatial state of stress we have derived

$$\Lambda_{\text{1D}} = \frac{\sigma_0^2}{2E} \text{ and } \Lambda_{\text{3D}} = \frac{1}{2E}\left[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\mu(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x)\right] + \frac{1}{2G}\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right). \qquad \text{… (FT\_5)}$$

Comparing them we obtain the equivalent stress in the form

$$\sigma_0 = \sqrt{2E\Lambda_{\text{3D}}}. \qquad \text{(FT\_6)}$$

Beltrami (FT_7)

The failure arises for $\sigma_0 = \sigma_{\mathrm{Pt}}$, where $\sigma_0 = \sqrt{2E\Lambda_{3\mathrm{D}}}$,

and $\Lambda_{3\mathrm{D}} = \dfrac{1}{2E}\left[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\mu\left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x\right)\right] + \dfrac{1}{2G}\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)$,

and σ_{Pt} is the allowable tensile strength.

6.5. Mises's hypothesis of the deviatoric part of the strain energy density

The stress com ponents corresponding to the spatial state of stre ss could be divided into two parts.

- One of them causes th e change of shape of the elem ent only, w ithout influencing its volume.
- The other causes the change of volum e of the elem ent only, without influencing its shape.

Evidently, the stress tensor could be decomposed as follows

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{yz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} \sigma_x - p & \tau_{xy} & \tau_{yz} \\ \tau_{yx} & \sigma_y - p & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - p \end{bmatrix} + \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}. \qquad \text{(FT\_8)}$$

Terminology

Stress	=	deviatoric stress i.e. the change of shape, volume is conserved.	+	volumetric stress i.e. the change of volume, shape is conserved.

It should be reminded that for the change of volume we can write

$$\Delta V = \mathrm{d}x(1+\varepsilon_x)\,\mathrm{d}y(1+\varepsilon_y)\,\mathrm{d}z(1+\varepsilon_z) \approx \left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z\,. \qquad \text{(FT\_9)}$$

Neglecting the higher-order quantities, the relative change of volume is

$$\frac{\Delta V}{V} \approx \left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right). \qquad \text{(FT\_10)}$$

Now, the q uestion is. W hat value the qu antity p has to attain in o rder that the condition $\mathrm{d}V = 0$ is satisfied? Meaning – no change of volume. Evidently,

$$\left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right) = 0\,. \qquad \text{(FT\_11)}$$

From the Hooke's law, we get

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - p - \mu\sigma_y - \mu\sigma_z\right), \qquad \text{(FT\_12)}$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - p - \mu\sigma_z - \mu\sigma_x\right), \qquad \text{(FT\_13a)}$$

$$\varepsilon_z = \frac{1}{E}\left(\sigma_z - p - \mu\sigma_x - \mu\sigma_y\right). \qquad \text{(FT\_13b)}$$

And substituting into $\varepsilon_x + \varepsilon_y + \varepsilon_z = 0$ we obtain

$$\left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right) = \frac{1-2\mu}{E}\left(\sigma_x + \sigma_y + \sigma_z - 3p\right) = 0. \qquad \text{(FT\_14)}$$

From the last equation, the p value could be evaluated as

$$p = \frac{1}{3}\left(\sigma_x + \sigma_y + \sigma_z\right). \qquad \text{(FT\_15)}$$

6.5.1. A part of the strain energy density evoked by the volumetric stress

If the body is loaded by pressure only, then all the normal stress component are equal to p. The corresponding strain components are also identical. We might write

$$\varepsilon = \varepsilon_x = \varepsilon_y = \varepsilon_z. \qquad \text{(FT\_16)}$$

The Hooke's law – for all three directions – is

$$\varepsilon = \frac{1}{E}\left(p - \mu p - \mu p\right) = \frac{p}{E}\left(1 - 2\mu\right). \qquad \text{(FT\_17)}$$

For the strain energy density, we have derived

$$\Lambda = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z\right) + \frac{1}{2}\left(\tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right). \qquad \text{(FT\_18)}$$

In the case of the pressure loading, the shear stress components vanish and we can write

$$\sigma_x = \sigma_y = \sigma_z = p \qquad \text{(FT\_19)}$$

and also

$$\varepsilon = \varepsilon_x = \varepsilon_y = \varepsilon_z. \qquad \text{(FT\_20)}$$

Then, from Eq. (FT_18) we get

$$\Lambda_{\mathrm{vol}} = 3\frac{1}{2}p\varepsilon = \frac{1}{2}p3\varepsilon. \qquad \text{(FT\_21)}$$

Since all three normal strain components are equal, the relative change of volume is

$$\frac{\Delta V}{V} = \left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right) = 3\varepsilon \ . \qquad \text{(FT\_22)}$$

So,

$$\varLambda_{\text{vol}} = 3\frac{1}{2}p\varepsilon = \frac{1}{2}p3\varepsilon = \frac{1}{2}p\frac{\Delta V}{V}. \qquad \text{(FT\_23)}$$

Comparing with Eq. (FT_17) we get

$$\varLambda_{\text{vol}} = 3\frac{1}{2}p\varepsilon = 3\frac{1}{2}p\frac{p}{E}(1-2\mu) = \frac{3(1-2\mu)}{2E}p^2. \qquad \text{(FT\_24)}$$

But $p = \frac{1}{3}\left(\sigma_x + \sigma_y + \sigma_z\right)$, so finally the part of the stra in energy density corresponding to the volumetric part of the stress is

$$\varLambda_{\text{vol}} = \frac{3(1-2\mu)}{2E}p^2 = \frac{3(1-2\mu)}{2E}\frac{1}{9}\left(\sigma_x + \sigma_y + \sigma_z\right)^2 = \frac{(1-2\mu)}{6E}\left(\sigma_x + \sigma_y + \sigma_z\right)^2. \ \text{(FT\_25)}$$

We have already shown that the total strain energy density is

$$\varLambda = \frac{1}{2E}\left[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\mu\left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x\right)\right] + \frac{1}{2G}\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right). \qquad \text{(FT\_26)}$$

6.5.2. A part of the strain energy density evoked by the deviatoric stress

So, the part of the strain energy density corres ponding to the deviatoric part of the stress is given by the difference

$$\varLambda_{\text{dev}} = \varLambda - \varLambda_{\text{vol}}. \qquad \text{(FT\_27)}$$

After the rearrangement we get

$$\varLambda_{\text{dev}} = \frac{1+\mu}{3E}\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - \sigma_x\sigma_y - \sigma_y\sigma_z - \sigma_z\sigma_x\right) + \frac{1}{2G}\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right). \qquad \text{(FT\_28)}$$

The strain energy density for the deviatoric part of uni-axial strain com es from the previous relationship in the form

$$\varLambda_{\text{dev}}^{\text{1D}} = \frac{1+\mu}{3E}\sigma_0^2. \qquad \text{(FT\_29)}$$

So, from the point of view of the m aterial strength, the uni-axial and sp atial state of stresses are equivalent if their deviatoric strain energy densities are equal, i.e.

$$\varLambda_{\text{dev}}^{\text{1D}} = \varLambda_{\text{dev}}, \qquad \text{(FT\_30)}$$

So,

$$\frac{(1+\mu)\sigma_0^2}{3E}=\frac{1+\mu}{3E}\left(\sigma_x^2+\sigma_y^2+\sigma_z^2-\sigma_x\sigma_y-\sigma_y\sigma_z-\sigma_z\sigma_x\right)+\frac{1}{2G}\left(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2\right), \quad \text{(FT\_31)}$$

$$\sigma_0^2=\left(\sigma_x^2+\sigma_y^2+\sigma_z^2-\sigma_x\sigma_y-\sigma_y\sigma_z-\sigma_z\sigma_x\right)+\frac{3E}{(1+\mu)}\frac{1}{2G}\left(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2\right). \quad \text{(FT\_32)}$$

And realizing that $\frac{E}{2G}=(1+\mu)$, we finally get

$$\sigma_0^2=\left(\sigma_x^2+\sigma_y^2+\sigma_z^2-\sigma_x\sigma_y-\sigma_y\sigma_z-\sigma_z\sigma_x\right)+3\left(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2\right). \quad \text{(FT\_33)}$$

So, according to the Mises's hypothesis, the equivalent stress is

$$\sigma_0=\sqrt{\left(\sigma_x^2+\sigma_y^2+\sigma_z^2-\sigma_x\sigma_y-\sigma_y\sigma_z-\sigma_z\sigma_x\right)+3\left(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2\right)}. \quad \text{(FT\_34)}$$

Mises (FT_35)

The failure arises for $\sigma_0=\sigma_{\mathrm{Pt}}$, where
$\sigma_0=\sqrt{\left(\sigma_x^2+\sigma_y^2+\sigma_z^2-\sigma_x\sigma_y-\sigma_y\sigma_z-\sigma_z\sigma_x\right)+3\left(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2\right)}$,
and σ_{Pt} is the allowable tensile strength.

This hypothesis is based on the assumption that the material failure is due to the deviatoric stress component only, or by other words due to the deviatoric strain energy density. In literature, this hypothesis is also known under the abbreviation HMH – meaning Huber-Mises-Henckey hypothesis.

6.6. Plasticity conditions

The subject of plasticity is a topic, requiring an extensive full semester course. There are many theories available; here – just have a just feeling for the subject – we present the two simplest hypotheses only.

Tresca's condition (FT_36)

H. Tresca, after extensive experiments, came to the conclusion that the plastic deformation of metals occurs if the following condition is met

$\sigma_0=\sigma_{\mathrm{K}}$, where $\sigma_0=\sigma_1-\sigma_3=2\tau_{\max}$.

Mises's condition (FT_37)

The plastic deformation of metals occurs if the following condition is met

$\sigma_0 = \sigma_\mathrm{K}$, kde $\sigma_0 = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1}$,

or $\sigma_0 = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]}$.

For more details see [7], [14], [17], [18].

07_PS. The principle of superposition

7.1. Introduction

The principle is valid for linear elastic systems only. The linear systems are characterized by the following assumptions: small displacements, infinitesimal strains, the equilibrium of force and stress quantities is considered in the un-deformed geometry[1], and finally the validity of Hooke's law. Under these assumptions the quantities to be determined are the linear functions of the applied loads.

Consider a schematized bridge, depicted in Fig. PS_1, that is subjected to external loading by two forces, say F_1, F_2. Let's analyze the deflection at the location 3.

In linear cases, the deformations are proportional to the applied forces.

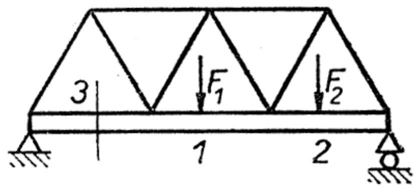


Fig. PS_1 ... Schematized bridge structure

We could thus state.

The deflection, at the location 3 due to the force F_1 alone, is $w_{31} = a_{31}F_1$.
The deflection, at the location 3 due to the force F_2 alone, is $w_{32} = a_{32}F_2$.

The quantities a_{31}, a_{32} are the proportionality constants. The principle of superposition states that the total deflection in the location 3 is

$$w_3 = w_{31} + w_{32} = a_{31}F_1 + a_{32}F_2 . \qquad \text{(PS\_1)}$$

It can be shown that the proportionality constants are functions of the structure's geometry only.

7.2. Betti's theorem[2]

Whether the acting forces are applied sequentially or all at once, the deformations of the analyzed structure are identical. Again, the theorem is valid for a "slow" application of forces and for linear elastic structures.

Imagine that the system of applied forces and moments is arbitrarily partitioned into two groups. Say, I and II.

Due to the loading of forces and moments belonging to the first group, the strain energy A_{I} is evoked in the system. Due to the consequent loading of forces and moments belonging to the second group, the strain energy A_{II} is introduced. Furthermore, there is additional energy $A_{\mathrm{I,II}}$

[1] This might happen if the large deformations occur. Then, due to the loading the initial geometry is substantially changed. This phenomenon is called the geometrical non-linearity.
[2] Enrico Betti Glaoui (1823 – 1892) was an Italian mathematician.

– this energy corresponds to the mechanical work done by forces and moments of the first group due to deformations evoked by the forces and moments of the second group.

So, the total strain energy is

$$A = A_{\mathrm{I}} + A_{\mathrm{I,II}} + A_{\mathrm{II}}. \qquad \text{(PS\_2)}$$

If the order of the loading process is reversed, then

$$A = A_{\mathrm{II}} + A_{\mathrm{II,I}} + A_{\mathrm{I}}. \qquad \text{(PS\_3)}$$

The total energy cannot depend on he order of the loading process, so

$$A_{\mathrm{I,II}} = A_{\mathrm{II,I}}. \qquad \text{(PS\_4)}$$

Summary for Betti's theorem

If a linear system (for which the superposition principle is valid) is subjected to the forces and moments belonging to two groups, then

– the mechanical work done by the first group of forces and moments due to the deformations evoked by the second group of forces and moments

is identical to

– the mechanical work done by the second group of forces and moments due to the deformations evoked by the first group of forces and moments.

7.3 Maxwell's theorem

If the principle of superposition is valid, then the deformations (deflections and slopes) are described by linear functions of the loading (forces and moments).

The so called *influence coefficients* are defined as follows

By the set of integer variables, $k = 1,2,\cdots n$, we define the points of action of single forces and moments. By another set of variables, $j = 1,2,\cdots n$, we define the locations where the deformations (deflection and slope) are observed.

So the influence coefficient is defined as the deformation at the location j evoked by a unit loading applied at the location k.

They might be denoted as follows:

α_{jk} … deflection at the location j … evoked by a unit force applied at the location k,
β_{jk} … slope at the location j … evoked by a unit force applied at the location k,
γ_{jk} … deflection at the location j … evoked by a unit moment applied at the location k,
δ_{jk} … slope at the location j … evoked by a unit moment applied at the location k.

Example – influence coefficients
Given: Cantilever beam subjected to the force and moment loading depicted in Fig. PS_2.
Determine: The slope and deflection at the location 4.

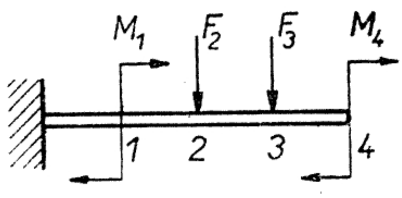


Fig. PS_2 ... Loaded beam

The deflection and the slope at the end of the beam are

$$w_4 = M_1\gamma_{41} + F_2\alpha_{42} + F_3\alpha_{43} + M_4\gamma_{44}, \qquad \text{(PS\_5)}$$
$$\phi_4 = M_1\delta_{41} + F_2\beta_{42} + F_3\beta_{43} + M_4\delta_{44}. \qquad \text{(PS\_6)}$$

Let's sort the loading effects into two groups as follows

$$\text{I}: \quad M_1, F_2, \qquad \text{(PS\_7)}$$
$$\text{II}: \quad M_4, F_3. \qquad \text{(PS\_8)}$$

Deflection evoked by the first group at the location 3, where F_3 acts, is $d_{\text{I}} = M_1\gamma_{31} + F_2\alpha_{32}$,
Slope evoked by the first group at the location 4 , where M_4 acts, is $s_{\text{I}} = M_1\delta_{41} + F_2\beta_{42}$,
Deflection evoked by the second group at the location 2, where F_2 acts, is $d_{\text{II}} = M_4\gamma_{24} + F_3\alpha_{23}$,
Slope evoked by the second group at the location 1,where M_1 acts, is $s_{\text{II}} = M_4\delta_{14} + F_3\beta_{13}$.

... (PS_9 to PS_12)

The mechanical work done by the first group due to the defor mations evoked by the second group is

$$A_{\text{I,II}} = M_1 s_{\text{II}} + F_2 d_{\text{II}}. \qquad \text{(PS\_13)}$$

The mechanical work done by th e second group due to the defo rmations evoked by the first group is

$$A_{\text{II,I}} = M_4 s_{\text{I}} + F_3 d_{\text{I}}. \qquad \text{(PS\_14)}$$

According to Betti's theorem these mechanical works (energies) are equal, so

$$A_{\mathrm{I,II}} = A_{\mathrm{II,I}} . \qquad \text{(PS\_15)}$$

Substituting and rearranging we get

$$M_1 F_3 (\beta_{13} - \gamma_{31}) + M_1 M_4 (\delta_{14} - \delta_{41}) + F_2 F_3 (\alpha_{23} - \alpha_{32}) + F_2 M_4 (\gamma_{24} - \beta_{42}) = 0 . \qquad \text{(PS\_16)}$$

In order that this equation be satisfied, the contents of its brackets have to be zero, so

$$\alpha_{jk} = \alpha_{kj}, \quad \beta_{jk} = \gamma_{kj}, \quad \delta_{jk} = \delta_{kj} . \qquad \text{(PS\_17)}$$

These equalities represent the Maxwell's theorem.

Summary for Maxwell's theorem

The deflection in the location j caused by the unit force in the location k is the same as the deflection in the location k caused by the unit force in the location j.

The slope in the location j caused by a unit force in the location k is the same as the slope in the location k caused by the unit moment in the location j.

For more details see [6], [7], [18], [19], [39].

08_PV. Principle of virtual work

8.1. Introduction

The *virtual work* is the mechanical work produced by forces and moments exerted during their virtual displacements. By the term *virtual displacement* we understand any infinitesimal displacement and/or rotation that are in agreements with the prescribed constraint conditions. For virtual quantities Lagrange introduced the symbol δ, to emphasize the virtual, i.e. fictional or apparent, character of these quantities. We assume that while the body is being transferred to a new, infinitesimally close position, the acting forces do not change their magnitudes and directions and that the time, during that transfer, is frozen.

8.2. Virtual work

In mechanics of deformable bodies the principle of virtual work states that the virtual work of internal forces , say δU, is equal to the virtual work of external forces, say δW, so

$$\delta W = \delta U \,. \qquad \text{(PV\_1)}$$

In mechanics of rigid bodies the deformations of loaded bodies are neglected, so the work done by internal forces is identically equal to zero, thus

$$\delta W = 0 \,. \qquad \text{(PV\_2)}$$

The rigid body in equilibrium is characterized by the fact that the resultant of all the forces and moments is identically equal to zero. If such a body is subjected to a virtual motion that is in agreement with constraints, then the resulting mechanical work, called the virtual work, is zero, as well. The condition of the zero virtual work is equivalent to the equilibrium condition.

At the first sight, the conclusion, that the zero resulting force produces zero work, seems to be trivial. But, the resulting zero is a sum of non-zero contributions of works produced by virtual displacements of individual forces. We will show that the power of the principle is based on the fact that the principle has to be valid for any virtual displacement.

The virtual displacements of deformable bodies are actually the displacements of individual particles representing the overall change of the body's shape – the rigid body motions are not considered. If a deformable body is in equilibrium, then the virtual work done by external forces is equal to the virtual work done by internal forces. The latter work is actually equal to the strain energy.

So,

$$\delta W - \delta U = \sum F_i \, \delta s_i - \delta U = 0 \,, \qquad \text{(PV\_3)}$$

where δs_i is a component of the virtual displacement of the point of action of the force F_i , having the same direction as the force F_i. It is assumed that the applied force does not change during the virtual displacement. The time is frozen, so the values of acting forces do not increase from zero to its final magnitude – that's why the factor $\frac{1}{2}$ does not appear in these expressions.

Often, the energy of external forces W, taken with the minus sign, is called the potential, i.e. $V = -W$. So, the energy W is capable to produce mechanical work, while the potential V consumes it. The strain energy is also capable to produce the mechanical work.

Introducing the total energy by

$$E = V + U \tag{PV\_4}$$

one can write

$$\delta E = -\delta A + \delta U = \delta V + \delta U = \delta(V + U) = 0\,. \tag{PV\_5}$$

This is, however, the condition for the minimum of the function W. It can be proved, see [22], that the condition of the minimum of the total energy is equivalent to the equilibrium condition.

In other words: The equilibrium conditions of a body occur for such a deformation configuration in which the minimum total energy is accumulated.

Example – the strain energy explained again

The linear elastic spring of the stiffness k is loaded by a force linearly increasing, i.e. $F = ku$, from zero to its maximum value F_{max}. Due to the loading, the length of the spring increases, see Fig. PV_1, and the accumulated strain energy is

$$\int F\,\mathrm{d}u = \int ku\,\mathrm{d}u = \frac{1}{2}ku^2 + C\,. \tag{PV\_6}$$

Generally, for a mechanical system characterized by displacements $\mathbf{u}$ and by the stiffness matrix $\mathbf{K}$, the strain energy expressed in the matrix form is

$$\frac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{K}\mathbf{u}\,. \tag{PV\_7}$$

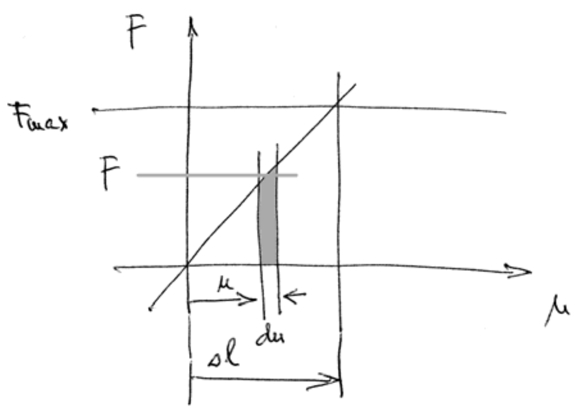


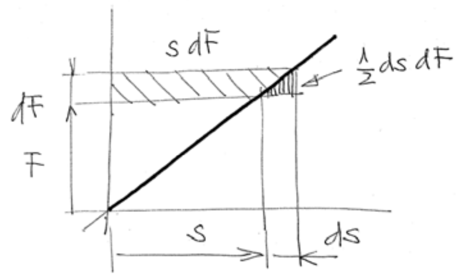
Fig. PV_1 ... Strain energy

We say that the strain energy is a quadratic function of displacements.

8.3. Castigliano's theorems

describe methods for determining the displacements of a linear elastic structure based on the partial derivatives of the energy.

Consider a body loaded by forces $F_1, F_2, \cdots, F_i$ and by moments $M_1, M_2, \cdots, M_j$. Let's analyze what happens if one of the forces, say F_k, $1 \le k \le i$ is changed by the increment $\mathrm{d}F_k$. Evidently, the strain energy increment is $\frac{\partial U}{\partial F_k}\mathrm{d}F_k$. Due to the applied force $\mathrm{d}F_k$ the loaded body deforms. Its point of action is displaced. Let the projection of this displacement be $\mathrm{d}s_k$. Then, the increment of the mechanical work has to be equal to the increase of the strain energy. The strain energy increments are depicted in Fig. PV_2. Thus,



$$\frac{1}{2}\mathrm{d}s_k\,\mathrm{d}F_k + s_k\,\mathrm{d}F_k = \frac{\partial U}{\partial F_k}\mathrm{d}F_k\,. \qquad \text{(PV\_8)}$$

Fig. PV_2 ... Strain energy increments

Neglecting the second order increments we obtain the *first Castigliano's theorem* in the form

$$s_k = \frac{\partial U}{\partial F_k}\,. \qquad \text{(PV\_9)}$$

A similar analysis, carried out for the moment quantities gives the *second Castigliano's theorem*.

$$\varphi_l = \frac{\partial U}{\partial M_l}\,. \qquad \text{(PV\_10)}$$

Example – application of the Castigliano's theorem

Given: Dimension, cantilever beam loaded by uniform distributed loading. See Fig. PV_3.
Determine: Deflection and slope as a function of the beam length.

A trick. At the free end of the beam a fictive force F and a fictive bending moment M_0 are added. After the analytical part of the solution is carried out, these quantities will be set to zero.

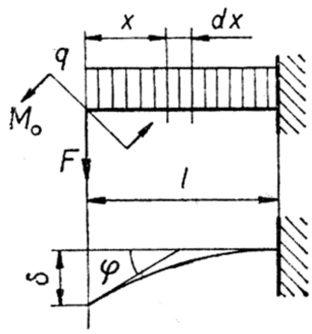


Fig. PV_3 ... Application of Castigliano theorem

The bending moment is

$$M(x) = -Fx - M_0 - \frac{qx^2}{2}. \qquad \text{(PV\_11)}$$

The strain energy accumulated in the beam is

$$U = \int_0^l \frac{M^2(x)}{2EJ_y} \mathrm{d}x. \qquad \text{(PV\_12)}$$

Applying the first Castigliano's theorem, we get the deflection at the free end of the beam in the form

$$\delta = \frac{\partial U}{\partial F} = \frac{\partial}{\partial F}\int_0^l \frac{M^2(x)}{2EJ_y}\mathrm{d}x = \int_0^l \frac{\partial}{\partial F}\frac{M^2(x)}{2EJ_y}\mathrm{d}x = \frac{1}{2EJ_y}\int_0^l 2M(x)\frac{\partial M(x)}{\partial F}\mathrm{d}x =$$
$$= \frac{1}{EJ_y}\int_0^l \left(-Fx - M_0 - \frac{qx^2}{2}\right)(-x)\mathrm{d}x \underbrace{\cdots\cdots}_{\text{substitute zeros}} \frac{1}{EJ_y}\int_0^l \left(-\frac{qx^2}{2}\right)(-x)\mathrm{d}x = \frac{ql^4}{8EJ_y}. \quad \ldots \text{(PV\_13)}$$

In the second line of the previous equation we have substituted zeros both for the fictive force and for the fictive moment.

Applying the second Castigliano's theorem, we get the slope at the end of the beam in the form

$$\varphi = \frac{\partial U}{\partial M_0} = \frac{1}{EJ_y}\int_0^l M(x)\frac{\partial M(x)}{\partial M_0}\mathrm{d}x = \frac{1}{EJ_y}\int_0^l \frac{qx^2}{2}\mathrm{d}x = \frac{ql^3}{6EJ_y}. \qquad \text{(PV\_14)}$$

Example

The Castigliano's theorem could be advantageously applied to solutions of statically indeterminate cases. See Fig. PV_4. Removing the left support and replacing it by the vertical reaction R, and realizing that there has to be the zero deflection there, we could write the Castigliano's theorem in the form

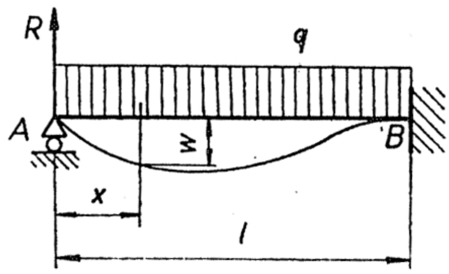


Fig. PV_4 ... Indeterminate beam

$$\frac{\partial U}{\partial R} = 0. \qquad \text{(PV\_15)}$$

The bending moment as a function of the x-coordinate is

$$M(x) = Rx - \frac{qx^2}{2}. \qquad \text{(PV\_16)}$$

The corresponding strain energy is

$$U = \frac{1}{2EJ_y}\int_0^l M^2(x)\mathrm{d}x = \frac{1}{2EJ_y}\int_0^l \left(R^2x^2 - Rqx^3 + \frac{q^2x^4}{4} \right)\mathrm{d}x = \frac{R^2l^3}{6EJ_y} - \frac{Rql^4}{8EJ_y} + \frac{q^2l^5}{40EJ_y}. \quad \text{(PV\_17)}$$

The derivative of the strain energy with respect to R has to be zero, so

$$\frac{\partial U}{\partial R} = \frac{2Rl^3}{6EJ_y} - \frac{ql^4}{8EJ_y} = 0, \quad \text{(PV\_18)}$$

which gives the unknown reaction

$$R = \frac{3ql}{8}. \quad \text{(PV\_19)}$$

The equation $\frac{\partial U}{\partial R} = 0$ could be viewed as the condition for the extreme of the function $U = U(R)$. We know that the condition for the extreme to become minimum requires that the second derivative of U with respect to R has to be positive. Evidently, in our example this condition is satisfied. Generally, the condition $\frac{\partial^2 U}{\partial R^2} > 0$ represents the so-called third Castigliano's theorem.

Expressed in words: Out of all the statically acceptable indeterminate reactions, the deformation condition is satisfied only for the reactions minimizing the strain energy.

This theorem is also known as the Menabrea's theorem.

8.4. Saint-Venant's principle

states that the difference between the effects of two different but statically equivalent loads is *very small* at a *sufficiently large* distance from the load.

This principle is used whenever it is necessary to idealize the task to be solved. Of course, the precise definitions of attributes *very small* and *sufficiently large* require a sound engineering insight.

As an example take the standard tensile test where we analyze the state of stress in the middle part of the test specimen specified by the uni-axial formula, i.e. $\sigma = F/S$, knowing that in locations where the specimen is clamped at its ends, there is a full 3D state of stress.

For more details see [17], [18], [19], [22], [39].

09_LM. Typical loading modes

9.1. Introduction to bending, torsion, and buckling

In the strength of material theory, the ability of various engineering machine parts to withstand the applied loading is treated by different approximate approaches – consequently, the different chapters are devoted to rods, beams, thin plates, thick plates, shells, thin-walled vessels, thick-walled vessels, etc. This scattered approach is – to certain respect avoided, when modern computational approaches – based on discretization (as for example the finite element method) are employed in practice. The author feels that knowledge of principles, on which the modern numerical methods are based, might be profitable for a student who intends to employ these methods efficiently. That's why the classical approaches for the treatment of bending, torsion, and buckling are presented in detail here.

9.2. Bending

9.2.1. Introduction

In engineering terminology the beam is a prismatic body being able to capture the external moments, the lateral and the longitudinal forces. Beams and their schematic representations are depicted in Fig. LM_1.

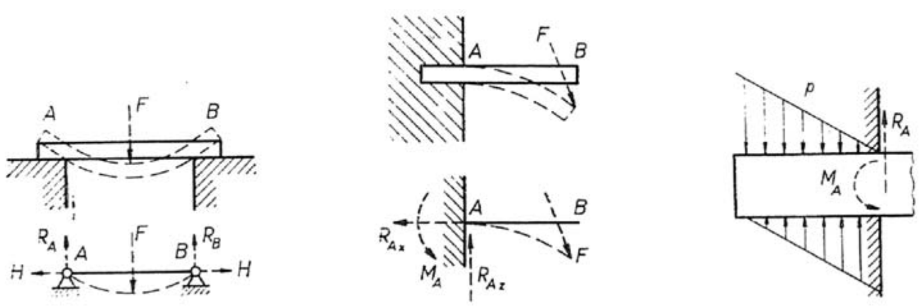


Fig. LM_1 … Beams and their schematic representations

9.2.1.1. Terminology

To remind the used terminology concerning the terms as the degrees of freedom, static determinacy, etc, the following text – already presented in the study of the mechanics of rigid bodies – is repeated here.

Different types of constraints – this subject was treated in detail the text devoted to mechanics of rigid bodies, so a brief repetition only – beams in the plane are treated here.

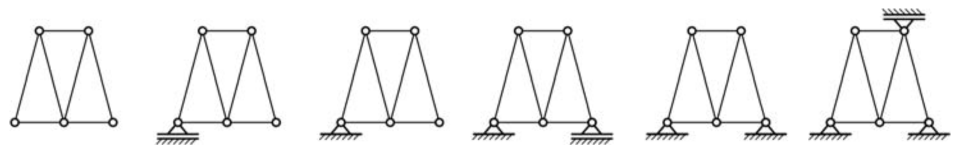
joint … radial – allows rotation, 2 reaction force components
… axiradial – allows rotation and displacement, 1 reaction force

clamping ... no displacements and rotations allowed, 2 reaction force components and 1 reaction moment

9.2.1.2. Explanation of terms: degrees of freedom, static determinacy, etc.

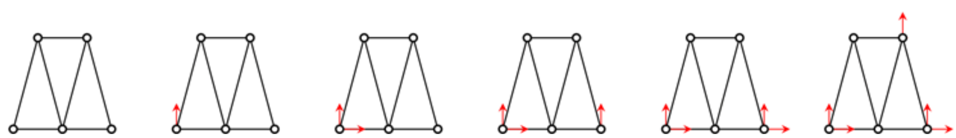
Six cases of a differently constrained body (a truss structure, composed of thin rods, also called bars) connected at their ends by frictionless joints, are depicted in Table LM_1. Due to miscellaneous constraints applied to that body, we can analyze six different cases with different numbers of degrees of freedom. For simplicity, the bridge structure is assumed to be two dimensional and all the constraints are considered frictionless. Two types of constraints are considered. First, a radial joint that besides the rotation allows left or right sliding motions. This constraint is also called a roller support. Second, a radial joint allowing a free rotation. This constraint is also called a pin support.

Structure



#dof's	3	2	1	0	-1	-2

reactions



# reactions comp.	0	1	2	3	4	5
# equilibrium eqs.	3	3	3	3	3	3
structure type	...... moving			properly constrained	... constrained ... too much	
type of problem	.... statically .... underdeterminate			statically determinate	... statically ... interdeterminate	
to be solved in	...... dynamics			statics	strength of material	

Table LM_1 ... Degrees of freedom and free body diagrams

The first column corresponds to a free, unconstraint or unsupported body that has 3 dof's in the plane. There are no reaction forces to be associated with the case.

The second column. The body is attached to the frame by a radial joint that besides the rotation allows left or right sliding motions. By mutual consent, the vertical motion in the up direction is prohibited. The body could freely rotate around the joint and also could freely move in left or right directions as well, it thus has two dof's. In the FBD this joint could be replaced by one unknown reaction component on the left, which would act vertically.

The third column. The body is attached to the frame by a radial joint allowing a free rotation around this joint only, it thus has one dof. In the FBD, this joint could be replaced by two unknown components of the reaction force in that joint.

The constraint bodies, depicted in the first three columns, have one common property, – they can move. Generally, the moving structures are characterized by the fact that their number of dof's is greater than zero. Mechanical systems composed of more rigid elements, having a positive number of dof's, are often called mechanisms. More about the subject is in the chapter devoted to kinematics.

Any structure able to move will start to change its position in space and cannot be treated by statics tools of mechanics. Their motions, due to the applied forces and moments, are described not by equations of equilibrium, but by equations of motions having the form of ordinary differential equations. In the following text, we will show how these problems are analyzed by tools of dynamics.

The fourth column. The body is attached to the frame at two places. On the left, there is a radial joint, which when considered alone, allows a free rotation. On the right, there is a sliding radial joint allowing both the rotation and the horizontal motions. The left joint removes one dof, and represents two unknown reaction components, the right one two dof's and requires to add one unknown reaction component in the FBD. Altogether, the body cannot move and has, in this case, zero degrees of freedom. Reaction forces represent three unknowns, two on the left and one on the right, and for a body in a plane, we have three scalar equations of equilibrium at our disposal. This case is thus easily solvable. We say that such a system is *statically determinate*.

Generally, we can state that the actual number of dof's of a body, say i, plus the number of unknown reaction components due to prescribed constraints, say m, is equal to the number of dof's of that body "freely" flying in the space (rigid body motions). In plane, we could write $i + m = 3$, in space $i + m = 6$.

The fifth and sixth columns correspond to structures that from the statics point of view are 'constrained too much'. They have a negative number of degrees of freedom. We say that these cases are *statically interdetermine*. In these cases, the number of unknown reaction components is greater than the number of available equilibrium equations. Consequently, the conditions of equilibrium do not suffice to find unknown reactions. Cases of this kind will be explained, analyzed and treated in chapters devoted to the mechanics of deformable bodies. We will show that adding an adequate number of so-called deformation conditions, the tasks of this type can be solved.

The treated tasks could be classified according to the number of degrees of freedom.

If # dof's = 0, then the mechanical system is said to be *statically determinate* and for given forces and moments, the corresponding reactions are readily obtained from properly formulated equilibrium conditions. In this case, the system is stationary and the number of unknowns is equal to the number of available equilibrium conditions.

If # dof's > 0, then the system is *statically underderterminate* and generally cannot be solved by statics tools. For given forces and moments, the system would start to move with accelerations and could only be treated by dynamics tools. Still, the tasks of this kind could be analyzed in statics if the problem is reformulated.

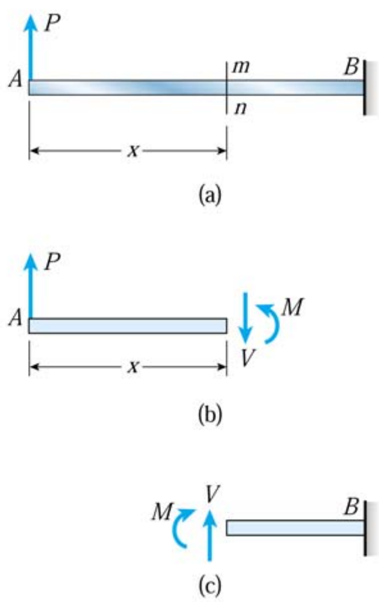
There are two possibilities.

First, for a given position determine such forces and moments that allow the system to stay in its current configuration.

Second, determine such a configuration in which the system – for a sufficient number of prescribed loads – will be in the state of equilibrium.

If # dof's < 0, then the system is said *statically indeterminate* and cannot be solved by statics tools since the number of unknown reactions is greater than the number of available equilibrium equations. The tasks of this kind could be treated by tools of mechanics of deformable bodies, where a suitable number of so-called deformation conditions are added, which together with equilibrium equations will suffice to find all the unknown reactions.

Internal actions (i.e. the shear force a nd the bending moment) in a cross section – again this was treated in detail in the text devoted to mechanics of rigid bodies



A cross section m-n divides the beam into two parts. See Fig. LM_2.

The left part. In this cross section, the indicated shear force V and the bending moment M replace the effects of the removed part. In this case, there are no forces in the lateral direction of the beam.

The right part. According to the principle of action and reaction, the forces and the moments acting on the right part are of the same magnitude but of opposite directions.

Fig. LM_2 … Free body diagram

To determine the type of reactions the free-body-diagram reasoning is used. To evaluate the magnitudes of reactions, the equilibrium conditions have to be applied and solved for reactions. Then, the internal forces and moments are determined from the condition of the equivalence of internal and external forces with reactions. For planar cases, two force components equations and one moment equation are required. Each component equation could be replaced by a moment equation. But not vice versa.

9.2.3. Sign conventions

The shear forces are considered positive if the material element turns clockwise. See Fig. LM_3.

The bending moment is considered positive if the upper part of the element is shortened, while the lower one is elongated.

We also accept the statement that the positive shear force and the positive bending moment deform the beam in a 'downward' fashion.

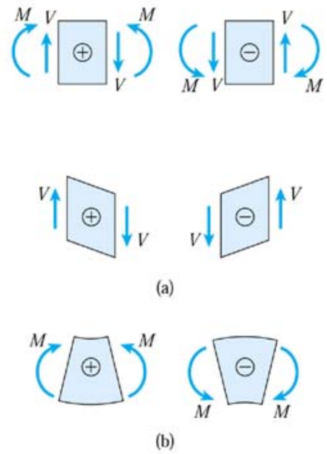


Fig. LM_3 … Sign convention

Example – simply supported beam loaded by two concentrated forces

Given: dimensions, forces. See Fig. LM_4.
Determine: the distributions of shear forces and the bending moments.

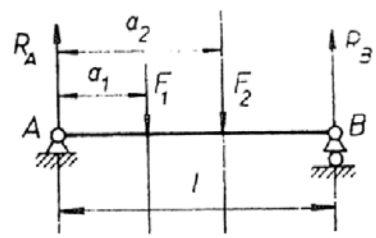


Fig. LM_4 … Simply supported beam forces

The considered beam is simply supported and thus statically determinate. Reactions are found from the equilibrium conditions. In plane, two component force and one moment equations are required. Since there are no axial forces applied, just two moment equilibrium equations are sufficient for the task. Solving them we get

$$R_{\mathrm{A}} = \frac{1}{l}\left[F_1(l - a_1) + F_2(l - a_2)\right], \quad R_{\mathrm{B}} = \frac{1}{l}\left[F_1 a_1 + F_2 a_2\right]. \qquad \text{(LM\_1)}$$

How do we proceed? At first, we cut the beam in a chosen cross section and apply the internal forces and moments in such a way that the overall equilibrium is assured. Then, the free-body diagram for internal, external and reaction forces and moments in the chosen cross section is constructed and finally, the equilibrium conditions for unknown internal forces and moments are solved. The equilibrium equations are considered for each part of the beam. Regardless of the chosen part of the beam, i.e. the left or right one, the considered equilibrium conditions should lead to the same results in terms of internal effects.

Now, back to our example.

The internal forces for the first part of the beam, i.e. for $0 \le x \le a_1$.

The removed part of the beam is replaced by forces and moments satisfying the overall equilibrium. See Fig. LM_4. Notice, that the principle of action and reaction is observed. To satisfy the equilibrium conditions, we added a vertical force T and a moment M. In this case, there is no internal horizontal force since no external forces are acting in that direction. The force T is called the *shear force*, while the moment M is called the *bending moment*.

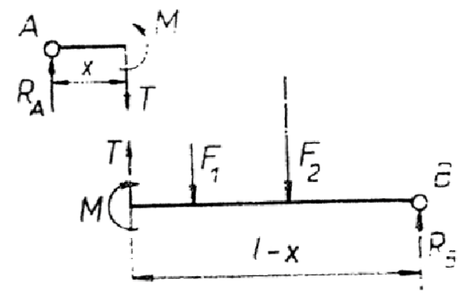


Fig. LM_4 … Simply supported beam forces first part

In the first part of the beam, i.e. for $0 \le x \le a_1$, the equilibrium conditions, written for the left-hand side of the beam, lead to

$$T = R_A, \quad M = R_A x. \tag{LM\_2}$$

Similarly, for the internal forces in the second part of the beam, i.e. for $a_1 < x < a_2$, see Fig. LM_5, we get

$$T = R_A - F_1, \quad M = R_A x - F_1 (x - a_1). \tag{LM\_3}$$

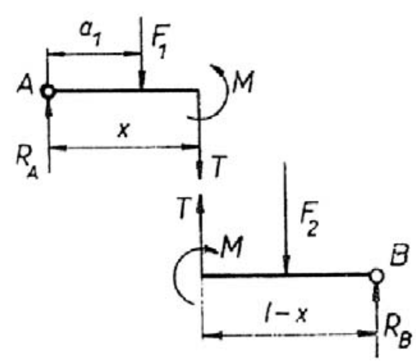


Fig. LM_5 … Simply supported beam forces second part

The internal forces in the third part of the beam, i.e. for $a_2 < x < l$. See Fig. LM_5.

Here, we can show that regardless of considering the left or the right part of the beam we get the same results. So, for the left-hand part we get

$$T = R_A - F_1 - F_2, \quad M = R_A x - F_1 (x - a_1) - F_2 (x - a_2), \tag{LM\_4}$$

while for the right-hand part we have

$$T = -R_B,$$
$$M = R_B(l - x). \quad \text{(LM\_5)}$$

When these, seemingly different, expressions are evaluated, they provide the same numerical results. In practice, it is recommended to consider that part of the beam, which requires the less effort for finding the result.

The distribution of the shear forces and of the bending moments as functions of the longitudinal coordinate is graphically depicted in Fig. LM_6.

The shown procedure for finding internal forces was already described in the text devoted to the mechanics of rigid bodies. Here, it is repeated for the self-consistency of the text. This way, we have obtained information about the distributions of internal shear forces and internal bending moments. So far, we know nothing about the deformations and stresses of the beam. This will be treated in the next paragraphs.

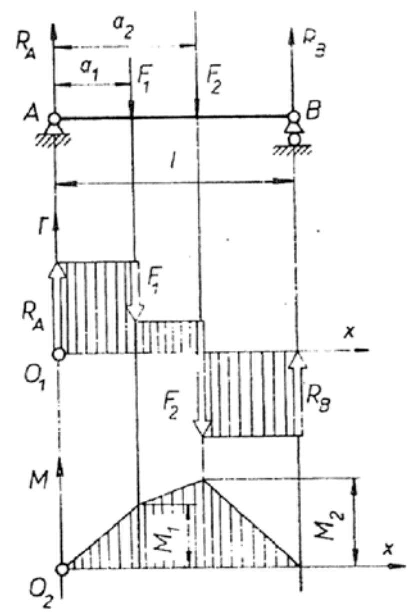


Fig. LM_6 … Simply supported beam forces

Example – simply supported beam with a uniformly distributed load of constant intensity

Given: dimensions, uniformly distributed loading q [N/m]. See Fig. LM_7.

Determine: Distribution of shear forces and bending moments.

This kind of distributed load might represent the loading due to the own weight, the layer of sand or snow, etc. It is measured in [N/m]. Due to the loading symmetry, we get the reactions by inspection

$$R_A = R_B = \frac{ql}{2}. \tag{LM\_6}$$

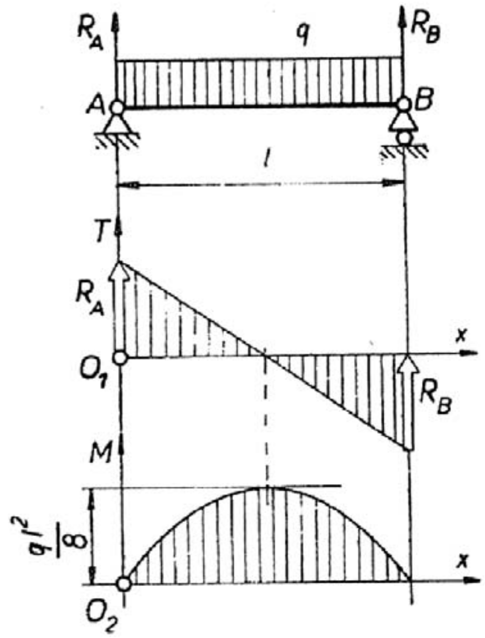


Fig. LM_7 … Simply supported beam distributed loading 1

Considering the equivalence of forces in the left part of the beam at a generic distance x, see Fig. LM_8, we get

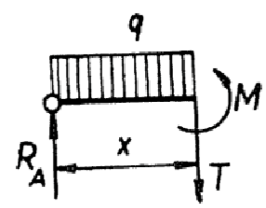


Fig. LM_8 … Simply supported beam distributed loading 2

$$T = R_A - qx = \frac{ql}{2} - qx = \frac{1}{2}q(l-2x),$$
$$M = R_A x - qx\frac{x}{2} = \frac{1}{2}qx(l-x). \quad \text{... (LM\_7)}$$

The distribution of the shear forces and the bending moment is depicted in Fig. LM_7.

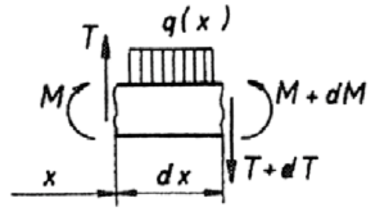
The maximum bending moment occurs in the middle of the beam, i.e. for $x = l/2$, and is

$$M_{\max} = M\big|_{x=l/2} = \frac{1}{8}ql^2. \tag{LM\_8}$$

Notice that the maximum bending moment corresponds to the zero shear force. Generally, it holds

$$T(x) = \frac{\mathrm{d}M(x)}{\mathrm{d}x}. \tag{LM\_9}$$

This relation can be proved in the following way. In Fig. LM_9, there is depicted a beam element of the length $\mathrm{d}x$ being subjected to a distributed load $q(x)$. Then for the shear forces, neglecting the increments of higher orders, we can write the equilibrium condition in the form



$$T + \mathrm{d}T + q\mathrm{d}x - T = 0 \qquad \text{(LM\_10)}$$

Fig. LM_9 … Schwedler

from which we get

$$q = -\frac{\mathrm{d}T}{\mathrm{d}x}. \qquad \text{(LM\_11)}$$

The moment equation of equilibrium, written with respect to the centre of the element, is

$$T\frac{\mathrm{d}x}{2} + M + (T + \mathrm{d}T)\frac{\mathrm{d}x}{2} - (M + \mathrm{d}M) = 0. \qquad \text{(LM\_12)}$$

From this equation we can deduce that

$$T = \frac{\mathrm{d}M}{\mathrm{d}x}. \qquad \text{(LM\_13)}$$

It is worth remembering that **the shear force is the first derivative of the bending moment while the distributed load is the negative derivative of the shear force**, thus

$$T(x) = \frac{\mathrm{d}M(x)}{\mathrm{d}x} \text{ and } q(x) = -\frac{\mathrm{d}T(x)}{\mathrm{d}x}. \qquad \text{(LM\_14)}$$

These relations were derived by J.W. Schwedler (1823 – 1894). In European textbooks, devoted to the subject of the engineering strength of material theory, they are known under the name of the Schwedler's theorems.

9.2.4. Deformations and stresses in beams subjected to pure bending

After determining the internal actions (shear forces and bending moments) in beams, we can proceed to establish the strains and stresses occurring due to the applied loading.

We will concentrate on prismatic beams living in the plane (xz) as depicted in Fig. LM_10.

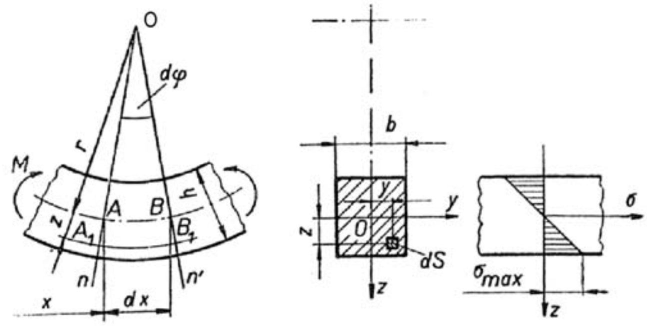


Fig. LM_10 … Beam_defo

The x-axis is positive in the 'right' direction, the positive z-axis is oriented 'downwards', while the positive y-axis is perpendicular to the plane (xz) and is directed 'to the viewer'. The beam is considered symmetric in the (xz) plane.

All the loads are assumed to act in the (xz) plane. If, futhermore, the cross-sectional area is symmetric with respect to (xz) plane, than the beam deflection occurs in the same plane – called the *plane of bending*. The initially straight longitudinal axis of the beam is bent – after the deformation it is called the *deflection curve* and is depicted by the dashed line. The normals to the deflection curve at points A and B intersect at the point O which is called the *center of curvature*. The indicated distance r is the *radius of curvature*, while its reciprocal value, i.e. $\frac{1}{r}$, is called the *curvature*.

Assume, that the beam is loaded by a moment M only. The cross-sectional area of the beam is $S = b \times h$. The prescribed loading of the beam evokes a deformation – its upper fibers are shortened, while the lower ones are elongated. Evidently, there must be a part of the beam cross section that is not deformed at all, it is called the *neutral surface* – its section with the (xz) plane is called the *neutral axis*.

Due to the deformation, the internal stress $\sigma(y,z)$ in the cross-sectional area S arises. The elementary force in the element of area $\mathrm{d}S$ is $\mathrm{d}N = \sigma(y,z)\mathrm{d}S = \sigma_x \mathrm{d}S$.

Since there are no other forces acting in the x-direction, the condition of equilibrium requires that the sum of all the elementary forces has to be equal to zero, thus

$$N = \int \mathrm{d}N = \int_S \sigma(y,z)\,\mathrm{d}S = 0. \qquad \text{(LM\_15)}$$

The moment of elementary internal forces $\mathrm{d}N$ with respect to the y-axis has to be equal to the external bending moment M . This moment is actually the magnitude of the moment vector $\vec{M} = \vec{M}_y$, i.e. $M = M_y = \left|\vec{M}_y\right|$, which is perpendicular to the cross-sectional area. Equilibrium condition requires that

$$M_y = \int z \, \mathrm{d}N = \int_S z \, \sigma(y,z) \, \mathrm{d}S. \tag{LM\_16}$$

Since we consider the in-plane bending only, the moment $\vec{M}_z$, whose magnitude is M_z, have to be zero. Thus,

$$M_z = \int y \mathrm{d}N = \int_S y\sigma(y,z)\mathrm{d}S = 0. \tag{LM\_17}$$

The last three equations, i.e. Eqs. (LM_15), (LM_16) and (LM_17), do not suffice for the unique determination of the internal actions in a cross section. Additional assumptions have to be accepted.

One of the possibilities is based on the so-called Bernoulli's[1] hypothesis, which assumes that the infinitesimally close cross sections – that were planar before the deformation – remain planar after the deformation as well. This assumption leads to the approximate theory which is known under different names – the theory of slender beams, the Bernoulli hypothesis or Bernoulli-Navier hypothesis.

Observe Fig. LM_10 again. The length of the part of the fiber between the points A and B, at a distance of z bellow the neutral axis, and measured before the deformation, is $\overline{AB} = \mathrm{d}x = \mathrm{r}d\varphi$, where r is the *radius of curvature*. After the deformation, this length changes to $\overline{A_1B_1} = (r+z)\mathrm{d}\varphi$.

So, the corresponding strain in the longitudinal direction is

$$\varepsilon(y,z) = \varepsilon_x = \frac{\overline{A_1B_1} - \overline{AB}}{\overline{AB}} = \frac{(r+z)\mathrm{d}\varphi - r\mathrm{d}\varphi}{r\mathrm{d}\varphi} = \frac{z}{r}. \tag{LM\_18}$$

It is assumed that there is no interaction between the neighboring fibers. Assuming also the validity of Hooke's law the corresponding stress component in the longitudinal direction is

$$\sigma(y,z) = \sigma_x = E\varepsilon_x = E\frac{z}{r}. \tag{LM\_19}$$

So, the longitudinal strain and stress components, in a beam loaded by pure bending, vary linearly with the distance measured from the neutral surface.

It should be reminded that the *first moment of area* of the cross-sectional area evaluated with respect to the *y*-axis could be expressed by means of the magnitude of area S and its centroid coordinate z_{T} in the form $\int z dS = z_{\mathrm{T}} S$.

---

[1] Jacob Bernoulli, 1654 – 1705, born in Basel, Swirzerland. He studied theology, mathematics and astronomy. Bernoulli discovered the constant $e = \lim_{n\to\infty}(1+1/n)^n$, which is the base of natural logarithm.

Substituting Eq. (LM_19) into Eq. (LM_15) we get

$$\frac{E}{r}\int_S z\,\mathrm{d}S = \frac{E}{r} z_T S = 0 . \qquad \text{(LM\_20)}$$

From this condition, we conclude that $z_T = 0$. This condition also defines the plane where the components of the longitudinal stresses and strains are equal to zero. This way, the *neutral surface*, where the strains and stresses are zero, is defined. Another important conclusion is that the neutral axis passes through the centriod of the cross-sectional area.

Substituting Eq. (LM_19) into Eq. (LM_16) we get

$$M_y = M = \frac{E}{r}\int z^2\,\mathrm{d}S = \frac{E}{r} J_y , \qquad \text{(LM\_21)}$$

where the quantity $J_y = \int z^2\,\mathrm{d}S, \quad [\mathrm{m}^4]$ is called the *second moment of area* with respect to the *y*-axis. For the plane bending problems the index $_y$, used for the quantities M_y, J_y defined above, is often omitted.

Summarizing, the theory for slender beams is based on the Bernoulli's relation

$$\frac{1}{r} = \frac{M}{EJ_y} , \qquad \text{(LM\_22)}$$

which states that the beam curvature is linearly proportional to the bending moment M and inversely proportional to the product EJ_y, which is called the *bending stiffness*.

From Eq. (LM_17) one can deduce

$$\int yz\,\mathrm{d}S = J_{yz} = 0 , \qquad \text{(LM\_23)}$$

where the quantity J_{yz} denotes the *deviatoric moment of area* with respect to axes y, z .

The Eq. (LM_23) is a necessary condition defining the state of the plane bending. It states that the plane bending occurs only if the *z*-axis is the symmetry axis of the beam's cross-sectional area.

From Eqs. (LM_19) and (LM_22) one can deduce that the longitudinal stress in the beam cross section depends on the distance from the neutral axis. For the positive bending moment, the upper part of the cross-sectional area shortens while the lower one prolongs. So, the cross sections of a beam in the state of the pure bending are in the state of uniaxial stress

$$\sigma(y,z) = \sigma_x = \frac{M}{J_y} z . \tag{LM_24}$$

Often, we are interested in the maximum value of the longitudinal stress only. In this case

$$(\sigma_x)_{\max} = \sigma_{\max} = \frac{M}{J_y} |z_{\max}| . \tag{LM_25}$$

If the cross-sectional area is symmetric with respect to the *y*-axis, then $|z_{\max}| = z_{\max}$, and we can write

$$\sigma_{\max} = \frac{M}{W_0} , \tag{LM_26}$$

$$W_0 = \frac{J_y}{z_{\max}} , \tag{LM_27}$$

where we have introduced a new variable W_0, called the *bending section module of the area*. The moduli for various types of the frequently used beam cross sections are listed in textbooks for the engineer's convenience. See [17], [39].

9.2.5. The conditions for the safe applicability of the slender beam theory

It should be reminded that the Bernoulli's beam theory (or by other words the slender beam theory) is based on the assumption of the state of pure bending. In engineering practice, such a loading is practically impossible to achieve – almost always there is a shear loading component present.

It is known that the shear forces produce so-called warping of the cross sections (i.e. their out of plane distortions). Thus, the cross sections, being planar before the deformation, are warped after the deformation and a more complicated beam theory has to be used. See [18]. It was, however, shown that when the slender beam assumptions are observed, the results obtained this way are acceptable.

The main assumptions for the safe applicability of the Bernoulli's theory are

- the cross-section dimension has to be small with respect to length of the beam,
- the errors are more significant in the vicinity of supports.

9.2.6. The second moments of area – see Fig. LM_11.

The second moments of area with respect to axes y, z are

$$J_y = \int_S z^2 \, \mathrm{d}S \,, \qquad \text{(LM\_28)}$$

$$J_z = \int_S y^2 \, \mathrm{d}S \,. \qquad \text{(LM\_29)}$$

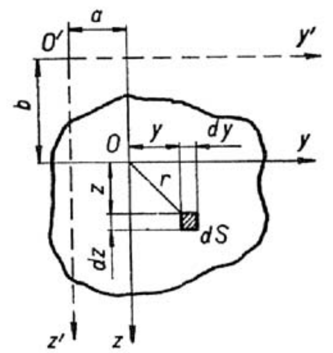


Fig. LM_11 … Moment of area 1

The deviatoric moment of area with respect to axes y, z is

$$J_{yz} = \int_S yz \, \mathrm{d}S \,. \qquad \text{(LM\_30)}$$

The polar moment of area is

$$J_p = \int_S r^2 \mathrm{d}S = \int_S \left(y^2 + z^2\right) \mathrm{d}S = J_z + J_y \,. \qquad \text{(LM\_31)}$$

The dimensions of these area moments are $[\mathrm{m}^4]$.

Example – the second moments of area for a beam with a rectangular cross section $b \times h$. See Fig. LM_12.

$$J_y = \int_S z^2 \mathrm{d}S = \int_{-b/2}^{b/2} \mathrm{d}y \int_{-h/2}^{h/2} z^2 \mathrm{d}z = \frac{1}{12} b h^3 \,, \qquad \text{(LM\_32)}$$

$$J_z = \int_S y^2 \mathrm{d}S = \int_{-h/2}^{h/2} \mathrm{d}y \int_{-b/2}^{b/2} y^2 \mathrm{d}y = \frac{1}{12} b^3 h \,, \qquad \text{(LM\_33)}$$

$$J_p = \int_S r^2 \mathrm{d}S = J_z + J_y = \frac{1}{12} b h \left(b^2 + h^2\right), \qquad \text{(LM\_34)}$$

$$J_{yz} = \int_S yz \mathrm{d}S = \int_{-b/2}^{b/2} y \mathrm{d}y + \int_{-h/2}^{h/2} z \mathrm{d}z = 0 \,. \qquad \text{(LM\_35)}$$

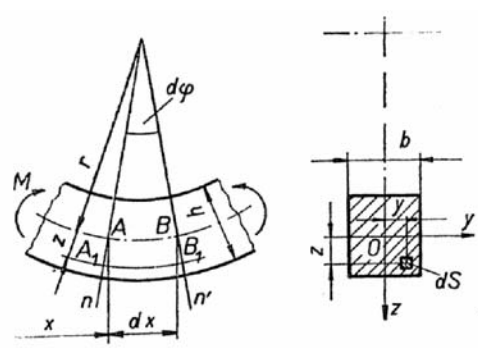


Fig. LM_12 … Beam defo 1

Example – the second moments for a circular cross section with the diameter d.

$$J_y = J_z = \frac{\pi}{64} d^4 \text{ and } J_p = \frac{\pi}{32} d^4 . \tag{LM\_36}$$

Example – the polar moment of area for an annulus with outer and inner diameters $D = 2r_2, d = 2r_1$ respectively

$$J_\mathrm{p} = \frac{\pi}{2}\left(r_2^4 - r_1^4\right) = \frac{\pi D^4}{32}\left(1 - \frac{d^4}{D^4}\right). \tag{LM\_37}$$

9.2.7. Parallel axis-theorem - the second moments of area with respect to shifted axes – see Fig. LM_11 again.

The second moments of area with respect to axes x', y' shifted by distances a, b are

$$J_{y'} = \int_S (z+b)^2 \mathrm{d}S = \int_S \left(z^2 + 2bz + b^2\right)\mathrm{d}S = \int_S z^2 \mathrm{d}S + 2b\int_S z\mathrm{d}S + b^2 \int_S \mathrm{d}S = J_y + 2bz_\mathrm{T}S + b^2 S , \tag{LM\_38}$$

where z_T is the area's centroid coordinate measured with respect to original axes and S is the cross-sectional area.

Similarly

$$J_{z'} = \int_S (y+a)^2 \mathrm{d}S = \int_S \left(y^2 + 2ay + a^2\right)\mathrm{d}S = \int_S y^2 \mathrm{d}S + 2a\int_S y\mathrm{d}S + b^2 \int_S \mathrm{d}S = J_z + 2ay_\mathrm{T}S + b^2 S ,$$

... (LM_39)

where y_T is the centroid coordinate measured with respect to original axes.

And finally, the deviatoric moment of area with respect to the shifted axes are

$$J_{y'z'} = \int_S (y+a)(z+b)\mathrm{d}S = \int_S yz\,\mathrm{d}S + a\int_S z\,\mathrm{d}S + b\int_S y\,\mathrm{d}S + ab\int_S \mathrm{d}S = J_{yz} + \left(az_\mathrm{T} + by_\mathrm{T}\right)S + abS .$$

... (LM_40)

If the original axes pass through the centroid of the cross-sectional area, i.e. $y_\mathrm{T} = z_\mathrm{T} = 0$, then the previous formulas simplify to

$$J_{y'} = J_y + b^2 S , \qquad J_{z'} = J_z + a^2 S , \qquad J_{y'z'} = J_{yz} + abS . \tag{LM\_41}$$

9.2.8. The second moments of area with respect to the rotated axes – see Fig. LM_13.

The relations between the original and the rotated coordinates are

$$\begin{aligned} \eta &= y\cos\alpha + z\sin\alpha \\ \zeta &= -y\sin\alpha + z\cos\alpha \end{aligned} \qquad \text{(LM\_42)}$$

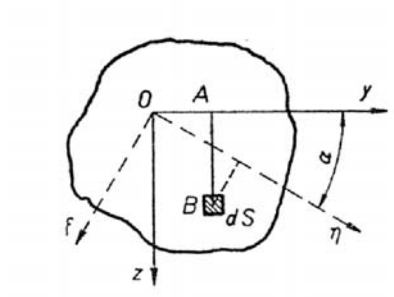


Fig. LM_13 … Turned axes

For the second moment of area with respect to η axis, one can write

$$J_\eta = \int_S \zeta^2 \mathrm{d}S = \int_S \left(y^2\sin^2\alpha - 2yz\sin\alpha\cos\alpha + z^2\cos^2\alpha\right)\mathrm{d}S = \\ = J_z\sin^2\alpha + 2J_{yz}\sin\alpha\cos\alpha + J_y\cos^2\alpha. \qquad \text{... (LM\_43)}$$

Similarly for the ζ axis

$$J_\xi = \int_S \eta^2 \mathrm{d}S = \int_S \left(y^2\cos^2\alpha - 2yz\sin\alpha\cos\alpha + x^2\sin^2\alpha\right)\mathrm{d}S = \\ = J_z\cos^2\alpha + 2J_{yz}\sin\alpha\cos\alpha + J_y\sin^2\alpha. \qquad \text{... (LM\_44)}$$

The deviatoric moment of area is

$$J_{\eta\zeta} = \int_S \eta\zeta\,\mathrm{d}S = -J_z\sin\alpha\cos\alpha + J_{yz}\left(\cos^2\alpha - \sin^2\alpha\right) + J_y\sin\alpha\cos\beta\,.$$

The above relations could be derived more efficiently in a matrix manner. Defining

$$\begin{bmatrix} J_\eta & J_{\eta\zeta} \\ J_{\eta\zeta} & J_\xi \end{bmatrix} = \begin{bmatrix} J'_{11} & J'_{12} \\ J'_{21} & J'_{22} \end{bmatrix} = \mathbf{J}' \;; \quad \begin{bmatrix} J_y & J_{yz} \\ J_{yz} & J_z \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \mathbf{J}\,; \quad \mathbf{R} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}. \qquad \text{... (LM\_45)}$$

we could simply write

$$\mathbf{J}' = \mathbf{R}^{\mathrm{T}}\mathbf{J}\mathbf{R}\,. \qquad \text{(LM\_46)}$$

In the beam theory there is often used another geometric quantity, i.e. the *bending section module of the area*, defined as

$$W_{\mathrm{o}} = \frac{J_y}{z_{\max}}. \qquad \text{(LM\_47)}$$

For the annulus with outer and inner diameters D and d respectively, we get

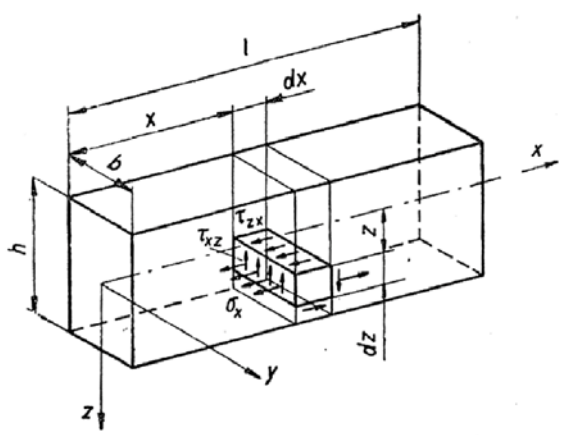
$$J_y = J_z = \frac{\pi}{64} D^4 - \frac{\pi}{64} d^4 \tag{LM\_48}$$

and then the *bending section module of the area* is

$$W_0 = \frac{J_y}{z_{\max}} = \frac{J_y}{D/2} = \frac{\pi}{32}\left(1 - \frac{d^4}{D^4}\right). \tag{LM\_49}$$

9.2.9. The influence of the shear force on the deformation of the beam

So far, only the influence of the pure bending moment was treated. Now, we will add analysis of the influence of the shear force. Let's consider a beam of the rectangular cross section $b \times h$ depicted in Fig. Fig. LM_14. As before, the bending moment is applied within the (*xz*) plane. The applied shear force is directed in the *z*-axis coordinate. The corresponding state of stress of an elementary prism is described by stress components shown in Fig. LM_15a.



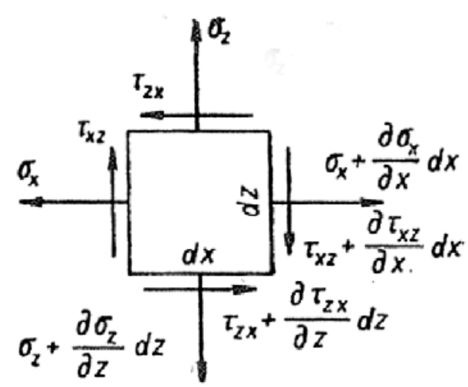


Fig. LM_14 … Beam_shear_1

Fig. LM_15a … Beam_shear_2

If the width of the cross-sectional area is sufficiently small, i.e. $b << h$, we might assume that the shear stress τ_{xz}, due to the shear force, is uniformly distributed along the beam's width, by other words it does not depend on the *y*-coordinate.

The equilibrium conditions written for x, z directions are

$$\left(\sigma_x + \frac{\partial \sigma_x}{\partial x}\right) b\mathrm{d}z - \sigma_x b\mathrm{d}z + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\right) b\mathrm{d}x - \tau_{zx} b\mathrm{d}x = 0, \tag{LM\_50}$$

$$\left(\sigma_z + \frac{\partial \sigma_z}{\partial z}\right) b\mathrm{d}x - \sigma_z b\mathrm{d}x + \left(\tau_{xz} + \frac{\partial \tau_{xz}}{\partial x}\right) b\mathrm{d}z - \tau_{xz} b\mathrm{d}y = 0. \tag{LM\_51}$$

Simplifying, we get

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{zx}}{\partial z} = 0, \tag{LM\_52}$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} = 0. \tag{LM\_53}$$

Substituting the relations derived for the pure bending, i.e.

$$\sigma_x = \frac{M(x)}{J_y} z \quad \text{and} \quad \sigma_z = 0 \tag{LM\_54}$$

and using the relation between the shear force and the bending moment

$$T(x) = \frac{\mathrm{d}M(x)}{\mathrm{d}x} \tag{LM\_55}$$

we get

$$\frac{\partial \tau_{zx}}{\partial z} = -\frac{z}{J_y} \frac{\mathrm{d}M(x)}{\mathrm{d}x)} = -\frac{T(x) z}{J_y}, \tag{LM\_56}$$

$$\frac{\partial \tau_{xz}}{\partial x} = 0.$$

Integrating Eq. (LM_56) we obtain

$$\tau_{zx} = -\frac{T}{J_y} \frac{z^2}{2} + C, \tag{LM\_57}$$

The unknown integration constant can be obtained form the condition of the free (unloaded) surface, i.e.

$$\tau_{zx}\big|_{z=\pm h/2} = 0. \tag{LM\_58}$$

So, $C = \dfrac{Th^2}{8J_y}$, (LM_59)

and finally

$$\tau_{zx} = \frac{T}{8J_y}\left(h^2 - 4z^2\right). \qquad \text{(LM\_60)}$$

For the rectangular area, where $J_y = \dfrac{1}{12}bh^2$, we get the shear stress as a quadratic function of the z-coordinate in the form

$$\tau_{zx} = \frac{3}{2}\frac{T}{bh}\left(1 - \frac{4z^2}{h^2}\right). \qquad \text{(LM\_61)}$$

The maximum shear stress value is for $z = 0$. So,

$$\left(\tau_{zx}\right)_{\max} = \frac{3}{2}\frac{T}{bh}. \qquad \text{(LM\_62)}$$

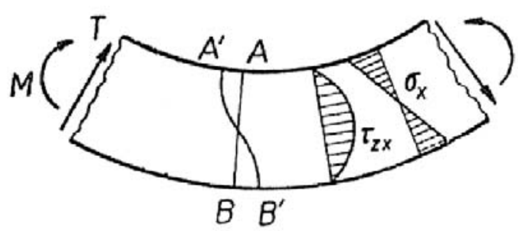


Fig. LM_15b … Beam_shear_3

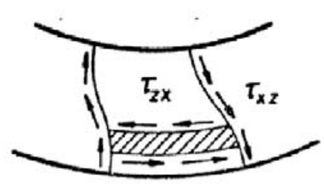


Fig. LM_15c … Beam_shear_4

The shear stresses have the parabolic appearance which is depicted in Fig. LM_15b. Due to the existence of shear stresses the initially planar surface AB is warped into the $A'B'$ shape. See Fig. LM_15c, where the influence of shear stresses is schematically indicated. This result, however, contradicts one of assumptions, which was accepted for the theory of beams being subjected to the state of pure bending.

So, the Bernoulli's pure bending theory presented above is approximate. But, still, it is useful in engineering computations. What are the limits of its validity is shown in the following example.

Example – validity of the pure bending theory

Given: Consider a cantilever beam, of the rectangular cross section $b \times h$, of the lenght l, loaded by a force F on its free end, i.e. $T = F = const$.
Determine: the relative errors due to neglecting the influence of shear forces.

In this case, the bending moment is a linear function of the beam's length and its maximum value is

$$\left|M_{\max}\right| = Fl \,. \tag{LM\_63}$$

According to Eq. (LM_62) we have

$$\left(\tau_{zx}\right)_{\max} = \frac{3}{2}\frac{T}{bh} \,. \tag{LM\_64}$$

Also, we have derived that $\sigma_{\max} = \dfrac{M}{W_0}$, $W_0 = \dfrac{J_y}{z_{\max}}$ and that for the rectangular area $b \times h$ $J_y = \dfrac{1}{12} bh^3$ and $z_{\max} = h/2$.

Putting it together we get

$$\left(\sigma_x\right)_{\max} = \frac{6Fl}{bh^2} = 4\frac{l}{h}\left(\tau_{zx}\right)_{\max} \,. \tag{LM\_65}$$

The ratio $(\tau_{zx})_{\max} / (\sigma_x)_{\max}$ for the varying ratio l/h is in Fig. LM_17.

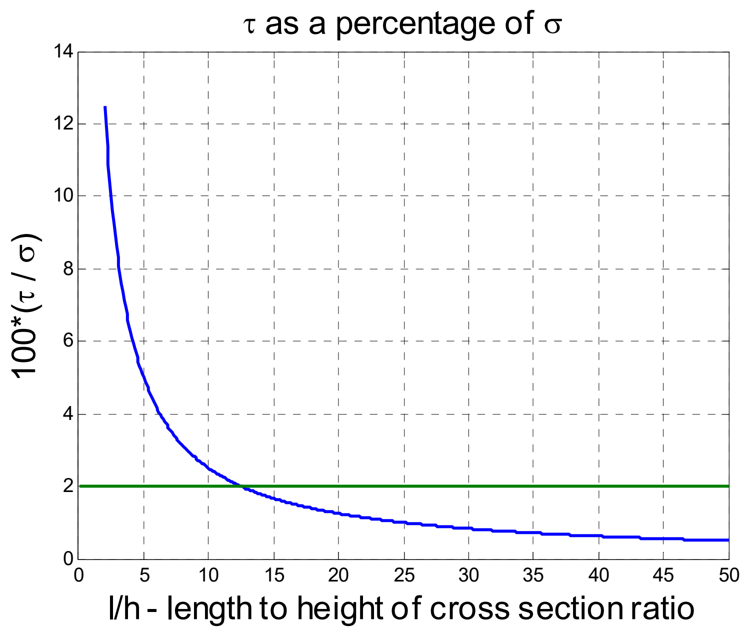


Fig. LM_17 … Shear to bending stress

For a particular beam considered in this example, the figure shows that the shear stress is less than two percents of the longitudinal one, provided that the beam's length is more than twelve times longer than the height of its cross-sectional area.

This conclusion might help to intuitive understanding what is the slender beam and under what conditions the influence of shear forces could be neglected.

The figure was created by the program **mpp_010e_beam_stress_ratio**

```
% mpp_010e_beam_stress_ratio
clear
lkuh = 2:0.1:50;
sigkutau = 4*lkuh;
taukusig = 1./sigkutau;
xx = [0 50];
yy = [2 2];

figure(1)
plot(lkuh, 100*taukusig, xx,yy, 'linewidth', 2); grid
xlabel('l/h - length to height of cross section ratio', 'fontsize', 16)
ylabel('100*(\tau / \sigma)', 'fontsize', 16)
title('\tau as a percentage of \sigma', 'fontsize', 16)
```

9.2.10. The differential equation of the deflection curve

We have stated that when analyzing the slender beams, the influence of the shear forces on deformations and stresses of could often be neglected and then the Bernoulli formula

$$\frac{1}{r} = \frac{M}{EJ_y},$$

could be employed. The curvature radius is r, the curvature $1/r$, the bending moment M, the elastic modulus is E and J_y is the second moment of the cross-sectional area. Often, the product EJ_y is constant and is called the *bending stiffness*.

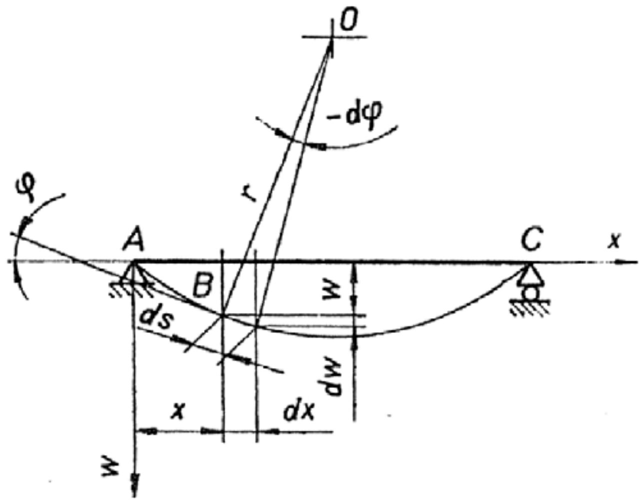


Fig. LM_18 … Bernoulli beam 01

Let the distance of a generic point B of the beam centre line from the left support be x, while the vertical deflection w is taken positively in the downward direction. See Fig. LM_18. We intend to find the function $w = w(x)$, called the *deflection curve* of the beam, describing the vertical deflection (often called displacements) as a function of the longitudinal distance x.

The normals of the deflection curve at locations x and $x + \mathrm{d}x$ intersect at the point O, which is the *local centre of curvature*.

The length of the indicated arc is

$$ds = -r d\varphi. \tag{LM\_66}$$

The minus sign indicates that with increasing length of the arc the tangent of the deflection curve diminishes.

It is obvious that

$$\tan\varphi = \frac{dw}{dx} = w' \text{ and } \mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}w^2. \tag{LM\_67}$$

The derivative of the inverse relation, i.e. $\varphi = \arctan w'$, with respect the elementary length, i.e. $ds = -r d\varphi$, gives

$$\frac{\mathrm{d}\varphi}{\mathrm{d}s} = \frac{\mathrm{d}\varphi}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{w''}{1+(w')^2}\frac{\mathrm{d}x}{\mathrm{d}s} = -\frac{1}{r}. \tag{LM\_68}$$

We could thus express the ratio

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\mathrm{d}x}{\sqrt{\mathrm{d}x^2 + \mathrm{d}w^2}} = \frac{1}{\sqrt{1 + \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2}} = \frac{1}{\sqrt{1 + (w')^2}} . \tag{LM\_69}$$

Substituting Eq. (LM_69) into Eq. (LM_68) we get the formula relating the curvature of the deflection curve to the x-coordinate.

$$\frac{1}{r} = -\frac{w''}{\left[1 + (w')^2\right]^{\frac{3}{2}}} . \tag{LM\_70}$$

Realizing that $\frac{1}{r} = \frac{M}{EJ_y}$, the differential equation of the deflection curve has the form

$$\frac{M}{EJ_y} = -\frac{w''}{\left[1 + (w')^2\right]^{\frac{3}{2}}} . \tag{LM\_71}$$

This equation is non-linear. In the linear theory elasticity the slope of the deflection curve, i.e. $\tan\varphi = \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)$, is small and could be approximated by the angle itself. Also the derivative value w' is small, thus the quadratic function $(w')^2$ is even smaller and could be neglected with respect to 1.

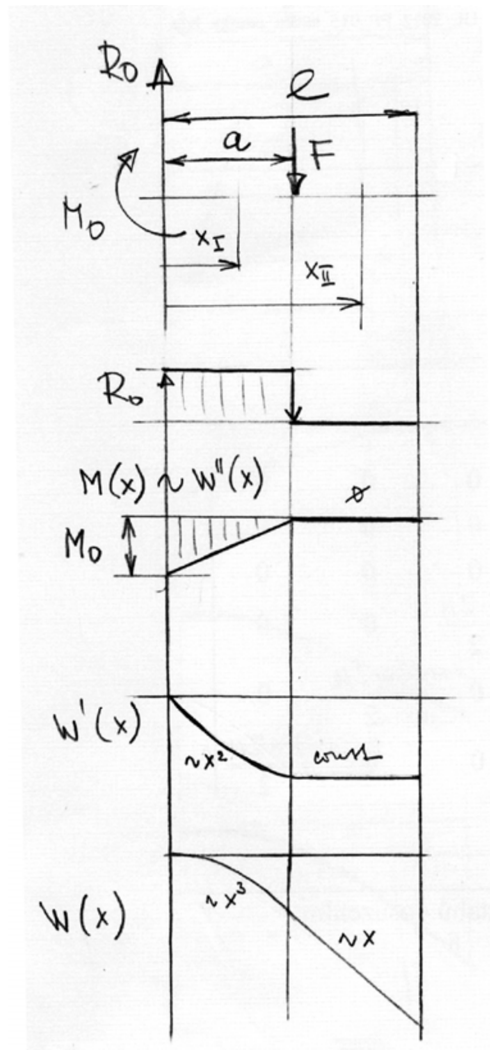
So, the simplified linear differential equation for the deflection curve is

$$w''(x) = \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} = -\frac{M(x)}{EJ_y} . \tag{LM\_72}$$

The minus sign on the right-hand side indicates that a positive bending moment introduces such a deflection that $w'' < 0$.

9.2.11. Beam examples

Example – Cantilever beam loaded by a force



Given: Cantilever beam clamped at its left part, dimensions, force F . Fig. LM_19.
Determine: Distribution of shear forces and bending moments along the beam.

Part I for $0 \le x \le a$

$M_{\mathrm{I}}(x) = -F(x-a)$ … bending moment, (LM_73),

$$w_{\mathrm{I}}''(x) = -\frac{M(x)}{EJ} = \frac{F(a-x)}{EJ}$$

… differential equation , (LM_74),

$$w_{\mathrm{I}}'(x) = \frac{Fa}{EJ}x - \frac{F}{2EJ}x^2 + C_1, \qquad \text{… slope, (LM\_75) ,}$$

$$w_{\mathrm{I}}(x) = \frac{Fa}{2EJ}x^2 - \frac{F}{6EJ}x^3 + C_1 x + C_2$$

… deflection, (LM_76).

Boundary conditions

$$w_{\mathrm{I}}'(0) = 0 \quad \Rightarrow \quad C_1 = 0, \qquad \text{(LM\_77)}$$
$$w_{\mathrm{I}}(0) = 0 \quad \Rightarrow \quad C_2 = 0. \qquad \text{(LM\_78)}$$

So, the slope and the deflection for $0 \le x \le a$ are

$$w_{\mathrm{I}}'(x) = \frac{F}{EJ}\left(ax - \frac{x^2}{2}\right), \qquad \text{(LM\_79)}$$

$$w_{\mathrm{I}}(x) = \frac{F}{EJ}\left(\frac{ax^2}{2} - \frac{x^3}{3}\right). \qquad \text{(LM\_80)}$$

Fig. LM_19 … Cantilever beam loaded by a force

Part II for $a \le x \le l$

$M_{\mathrm{II}}(x) = 0$ ... bending moment, (LM_81)
$w_{\mathrm{II}}(x) = 0$ ... differential equation, (LM_82)
$w_{\mathrm{II}}'(x) = C_3$ ... slope, (LM_83)
$w_{\mathrm{II}}(x) = C_3 x + C_4$ ... deflection. (LM_84)

Boundary conditions

$$w'_{\mathrm{I}}(a) = w'_{\mathrm{II}}(a) \quad \Rightarrow \quad C_3 = \frac{Fa^2}{2EJ},$$

$$w_{\mathrm{I}}(a) = w_{\mathrm{II}}(a) \quad \Rightarrow \quad C_3 a + C_4 = \frac{Fa^3}{3EJ}, \qquad \text{... (LM\_85)}$$

$$\Rightarrow \quad C_4 = -\frac{Fa^3}{6EJ}.$$

So, the slope and the deflection for $a \le x \le l$ are

$$w'_{\mathrm{II}}(x) = \frac{Fa^2}{2EJ}, \qquad \text{(LM\_86)}$$

$$w_{\mathrm{II}}(x) = \frac{Fa^2}{2EJ}x - \frac{Fa^3}{6EJ}. \qquad \text{(LM\_87)}$$

See the program **mpp_014e_cantilever_beam_single_force** and its graphical output in Fig. LM_20.

```
% mpp_014e_cantilever_beam_single_force
% which is loaded by a point force at
% the distance a from the clamped end

clear; format long e
l = 1;                    % beam length
a = 0.6;                  % position of F force, measured form the clamped end
b1 = 0.05; h1 = 0.05;     % dimensions of rectangular cross section
Jy = b1*h1^3/12;          % cross-sectional moment
F = 1000;                 % loading [N]
E = 2.1e11;               % Young modulus

incr = 0.01;              % 'x' variable increment
xrange = [0:incr:l];      % range of 'x' variable

const1 = F/(E*Jy);
const2 = F/(2*E*Jy);
const3 = F/(6*E*Jy);

ix = 0;
for x = xrange
    ix = ix + 1;
    if x <= a
        s(ix) = const1*(a*x - x^2/2);
        w(ix) = const1*(a*x^2/2 - x^3/6);
    else
        s(ix) = const2*a^2;
        w(ix) = const2*a^2*x - const3*a^3;
    end
end

xx = [a a];
yy = [0.05e-3 -4e-3];
figure(1)
plot(xrange,-w, 'k-', xrange,-s, 'k:', xx,yy, 'linewidth', 2)
title('cantilever beam, force applied at x=a', 'fontsize', 16)
xlabel('x-coordinate', 'fontsize', 16); grid
ylabel('slope and displacement', 'fontsize', 16)
legend('displacement', 'slope', 3)
```

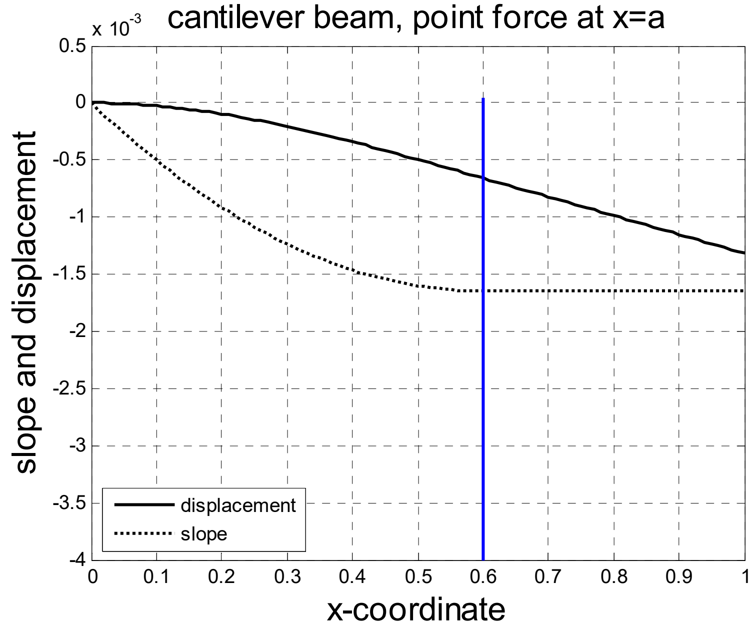


Fig. LM_20 … Slope and displacement for a cantilever beam loaded by a force

Example – a simply supported beam with a uniformly distributed load of constant intensity

Given: Dimensions, distributed load q $[\mathrm{N/m}]$.
See Fig. LM_21.
Determine: The deflection curve $w = w(x)$ and its slope $\varphi = w'(x)$.

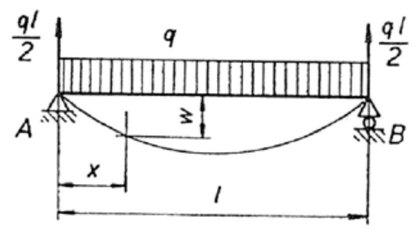


Fig. LM_21 … Simply supported beam cont load

There are no forces in the x-direction. Due to the symmetry, the vertical reactions are $R_A = R_B = \dfrac{ql}{2}$. At a generic point located in the distance x from the left-hand support, the bending moment is

$$M(x) = \frac{ql}{2}x - qx\frac{x}{2} = \frac{1}{2}qx(l-x). \qquad \text{(LM\_88)}$$

Substituting it into $w''(x) = -\dfrac{M(x)}{EJ_y}$ we get the second derivative of the deflection curve as a function of the x-coordinate in the form

$$w''(x) = -\frac{q}{2EJ_y}\left(lx - x^2\right). \tag{LM\_89}$$

The slope of the deflection curve and the vertical deflection of the beam are obtained by consecutive integrations of Eq. (LM_89) with respect to the *x*-coordinate.

$$w'(x) = -\frac{q}{2EJ_y}\left(l\frac{x^2}{2} - \frac{x^3}{3}\right) = -\frac{q}{2EJ_y}\left(\frac{3lx^2 - 2x^3}{6}\right) = -\frac{q}{12EJ_y}\left(3lx^2 - 2x^3 + C_1\right), \tag{LM\_90}$$

$$w(x) = -\frac{q}{24EJ_y}\left(2lx^3 - x^4 + 2C_1x + C_2\right). \tag{LM\_91}$$

Two unknown integration constants are obtained by satisfying the pertinent boundary conditions. For the simply supported beam, it is obvious that the vertical displacements (deflections) at the locations, where the beam is supported, have to be identically equal to zero.

So,

a) $w(0) = 0$, (LM_92)
b) $w(l) = 0$. (LM_93)

Substituting the condition add a) into Eq. (LM_91) and realizing that $x = 0$, we get

$$0 = -\frac{q}{24EJ_y}(0 - 0 + 0 + C_2) \qquad \Rightarrow C_2 = 0\,. \tag{LM\_94}$$

Substituting the condition given by Eq. (LM_92) into Eq. (LM_93) and realizing that $x = l$, we get

$$0 = -\frac{q}{24EJ_y}(2l^4 - l^4 + 2C_1 l) \qquad \Rightarrow C_1 = -\frac{l^3}{2}\,. \tag{LM\_95}$$

Substituting the obtained integration constants into Eqs. (LM_(90) and (LM_91) we get the slope and the deflection of a simply supported beam subjected to a distributed load q.

$$\varphi(x) = w'(x) = \frac{q}{24EJ_y}\left(l^3 - 6lx^2 + 4x^3\right), \tag{LM\_96}$$

$$w(x) = \frac{qx}{24EJ_y}\left(l^3 - 2lx^2 + x^3\right). \tag{LM\_97}$$

See the program **mpp_011e_beam_deflection_slope** and its output in Fig. LM_22.

```
% mpp_011e_beam_deflection_slope
clear
xrange = 0:0.05:1;
l = 1;
b = 0.05; h = 0.05;
Jy = b*h^3/12;
q = 1000; E = 2.1e11;
konst = q/(24*E*Jy);
i = 0;
for x = xrange
    i = i + 1;
    fi(i) = konst*(l^3 - 6*l*x*x + 4*x^3);
    w(i) = konst*x*(l^3-2*l*x^2 + x^3);
end

figure(1)
plot(xrange,fi,'k:', xrange,w,'k-', 'linewidth', 2 )
grid
legend('slope', 'deflection', 1)
```

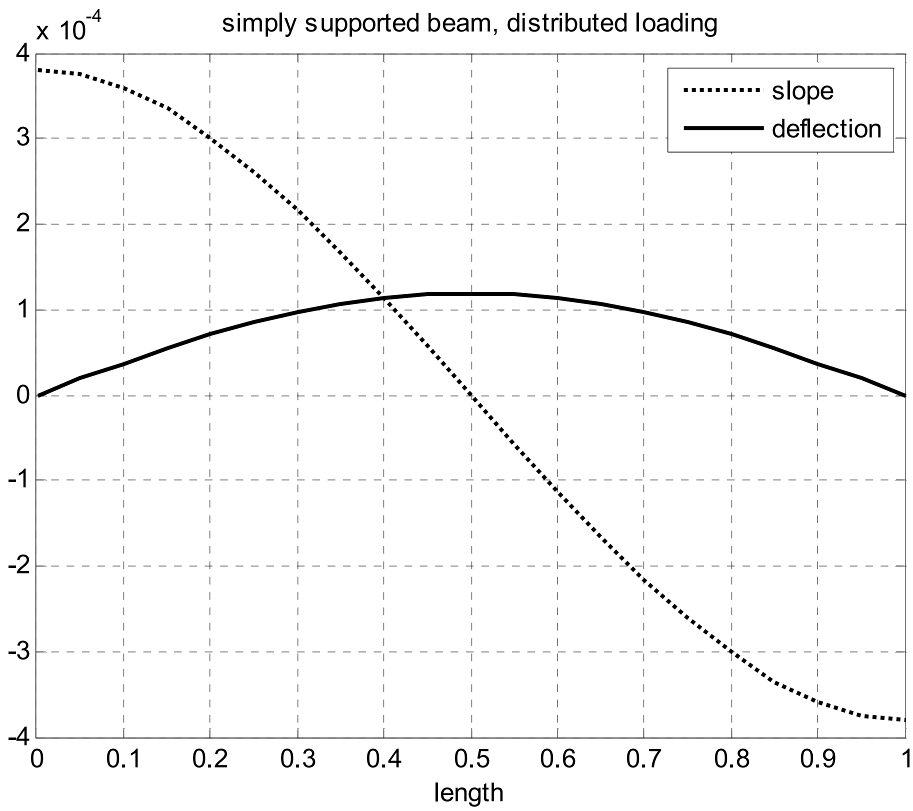


Fig. LM_22 … Slope and deflection of a simply supported beam subjected to a distribute load

Notice that in Fig. LM_21 the positive deflection of the beam plotted is in the upward direction. This is an ordinary tradition in mathematics, but in engineering texts oriented to beam treatments, the positive deflection is often oriented downwards.

Example – simply supported beam loaded by a single force

Given: Dimensions, force F . See Fig. LM_23.
Determine: The deflection curve $w = w(x)$ and its slope $\varphi = w'(x)$.

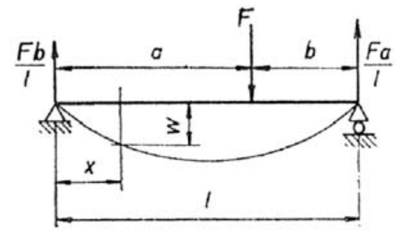


Fig. Fig. LM_23 … Simply supported beam force load

At first, the reactions have to be found. From the equilibrium conditions we obtain

$$R_{\mathrm{A}} = \frac{Fb}{l}, \quad R_{\mathrm{B}} = \frac{Fa}{l}. \qquad \text{(LM\_98)}$$

The bending moment is defined differently for the left-hand part of the beam, i.e. for $0 \le x \le a$, and for the right-hand part, i.e. for $a \le x \le l$.
For the left part, $0 \le x \le a$, the bending moment is

$$M_1(x) = \frac{Fb}{l} x, \qquad \text{(LM\_99)}$$

while for the right part, $a \le x \le l$, the bending moment is

$$M_2(x) = \frac{Fb}{l} x - F(x-a). \qquad \text{(LM\_100)}$$

By subsequent substitution of $M_1(x), M_2(x)$ into $w''(x) = -\dfrac{M(x)}{EJ_y}$ for both intervals we get two relations.

For $0 \le x \le a$

$$w_1''(x) = -\frac{M_1(x)}{EJ_y} = -\frac{Fb}{lEJ_y} x. \qquad \text{(LM\_101)}$$

For $a \le x \le l$

$$w_2''(x) = -\frac{M_2(x)}{EJ_y} = -\frac{Fb}{lEJ_y} x + \frac{F}{EJ_y}(x-a). \qquad \text{(LM\_102)}$$

Integrating Eq. (LM_101) twice we get

$$w_1'(x) = -\frac{Fb}{2lEJ_y}x^2 + C_1 \qquad \text{... slope1,} \qquad \text{(LM\_103)}$$

$$w_1(x) = -\frac{Fb}{6lEJ_y}x^3 + C_1x + C_2 \qquad \text{... deflection1.} \qquad \text{(LM\_104)}$$

Integrating Eq. (LM_102) twice we get

$$w_2'(x) = -\frac{Fb}{2lEJ_y}x^2 + \frac{F}{2EJ_y}(x-a)^2 + C_3 \qquad \text{... slope2,} \qquad \text{(LM\_105)}$$

$$w_2(x) = -\frac{Fb}{6lEJ_y}x^3 + \frac{F}{6EJ_y}(x-a)^3 + C_3x + C_4 \qquad \text{... deflection2.} \qquad \text{(LM\_106)}$$

Four unknown integration constants are determined from the boundary conditions.

The first two boundary conditions require that vertical deflections at locations where the beam is supported have to be zero. So,

$$w_1(0) = 0\,, \qquad \text{(LM\_107)}$$
$$w_2(l) = 0\,. \qquad \text{(LM\_108)}$$

The third boundary condition expresses the condition of the deflection continuity under the loading force, thus

$$w_1(a) = w_2(a)\,. \qquad \text{(LM\_109)}$$

The deflection has to be not only continuous but should be smooth as well. So, the first derivatives of deflection at $x = a$ from the left and from the right parts have to be equal. Thus, the fourth boundary condition is

$$w_1'(a) = w_2'(a)\,. \qquad \text{(LM\_110)}$$

Four boundary conditions mentioned above, suffice to determine four integration constants.

Eq. (LM_107) substituted into Eq. (LM_104) gives $C_2 = 0$.
From Eq. (LM_109) follows that $C_1 = C_3$.
From Eq. (LM_110), follows that $C_2 = C_4$. And thus, $C_4 = 0$.
Eq. (LM_108) substituted into Eq. (LM_106) (deflection2) gives

$$w_2(l) = -\frac{Fbl^2}{6EJ_y} + \frac{F}{6EJ_y}(l-a)^3 + C_3l = 0\,. \qquad \text{(LM\_111)}$$

Realizing that $l - a = b$ we finally get

$$C_3 = \frac{Fb}{6lEJ_y}\left(l^2 - b^2\right). \tag{LM\_112a}$$

The slope and deflection for $0 \le x \le a$ are

$$\varphi_1(x) = w_1'(x) = \frac{Fb}{6lEJ_y}\left(l^2 - b^2 - 3x^2\right), \tag{LM\_112b}$$

$$w_1(x) = \frac{Fbx}{6lEJ_y}\left(l^2 - b^2 - x^2\right). \tag{LM\_113}$$

The slope and the deflection for $a \le x \le l$ are

$$\varphi_2(x) = w_2'(x) = \frac{Fb}{6lEJ_y}\left(l^2 - b^2 - 3x^2\right) + \frac{F}{2EJ_y}(x-a)^2, \tag{LM\_114}$$

$$w_2(x) = \frac{Fbx}{6lEJ_y}\left(l^2 - b^2 - x^2\right) + \frac{F}{6EJ_y}(x-a)^3. \tag{LM\_115}$$

For a given value of b we could determine the maximum deflection due to the applied force F. The sought-after maximum of deflection is obtained from the condition $w_1'(x_{\max}) = 0$, thus

$$\frac{Fb}{6lEJ_y}\left(l^2 - b^2 - 3x_{\max}^2\right) = 0, \tag{LM\_116}$$

which gives the location of the maximum deflection for the given position of the force indicated by the distance b.

$$x_{\max} = \sqrt{\frac{1}{3}\left(l^2 - b^2\right)}. \tag{LM\_117}$$

For the symmetric loading, i.e. for $b = l/2$, we get

$$x_{\max} = \sqrt{\frac{1}{3}\left(l^2 - \frac{l^2}{4}\right)} = \frac{l}{2}. \tag{LM\_118}$$

Substituting $x_{\max} = \sqrt{\frac{1}{3}\left(l^2 - b^2\right)}$ into the Eq. (LM_113), we get the maximum value of the deflection as a function of b in the form

$$w_{\max} = w_1(x_{\max}) = \frac{Fb\left(l^2 - b^2\right)^{\frac{3}{2}}}{9\sqrt{3}EJ_y l}\,. \qquad \text{(LM\_119)}$$

For the force F approaching the right support, i.e. for $b \to 0$, the location of the maximum deflection will approach the value

$$x_{\max}^{\text{limit}} = \lim_{b \to 0} x_{\max} = \sqrt{\frac{1}{3}}\, l \approx 0{,}577\, l, \qquad \text{(LM\_120)}$$

which is not very distant from the centre of the beam. It is a little bit surprising. Finally, let's determine the deflection for the force acting just in the middle of the beam. Substituting $x = l/2$ into Eq. (LM_113) we get

$$w_1\left(\frac{l}{2}\right) = \frac{Fb\dfrac{l}{2}}{6lEJ_y}\left(l^2 - b^2 - \frac{l^2}{4}\right) = \frac{Fb}{48EJ_y}\left(3l^2 - 4b^2\right). \qquad \text{(LM\_121)}$$

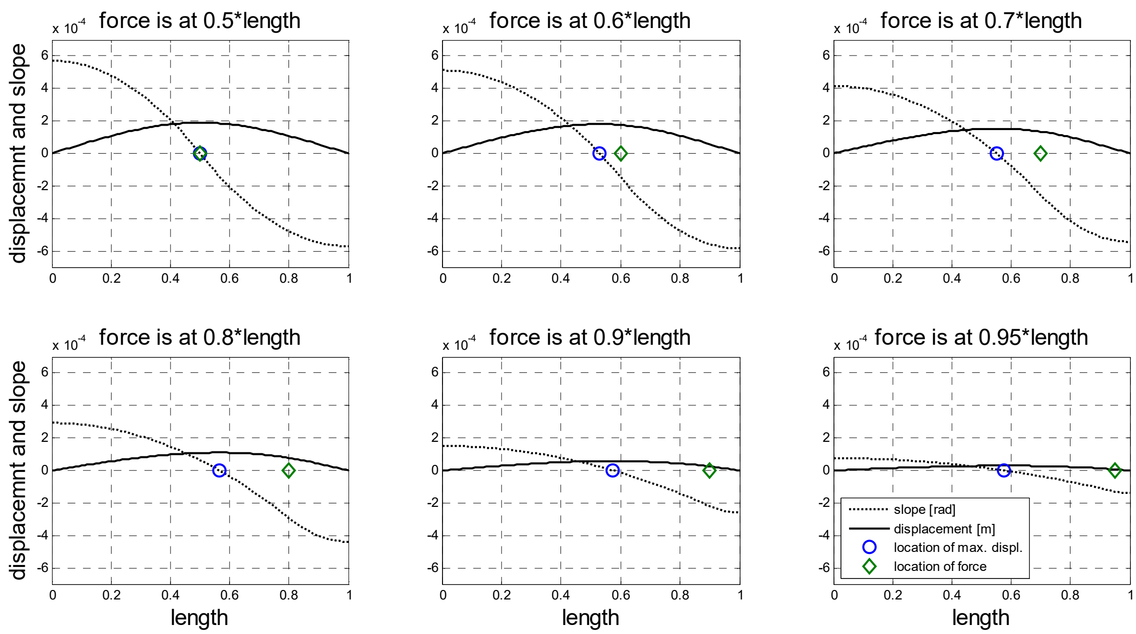


Fig. LM_24 … Simply supported beam – deflection and slope for a varying location of the force

In this case, however, $b = \dfrac{l}{2}$, so we finally get the often used formula for the deflection of the beam being loaded in the middle in the form

$$w_1\left(\frac{l}{2}\right) = \frac{Fl^3}{48EJ_y}\,. \qquad \text{(LM\_122)}$$

In Fig. LM_24 there are depicted deflections and slopes along the length of the beam for a varying location of the applied force. The series of subplots show the subsequent 'motion' of the force from the middle part of the beam to its right support. The location of the force is indicated by a diamond, while the location of the corresponding maximum deflection is indicated by a circle. An interesting observation: The maximum deflection does not occur under the applied force.

Fig. LM_25 shows the locations of the maximum beam deflections as a function of the force location. Only the second part of the beam is treated.

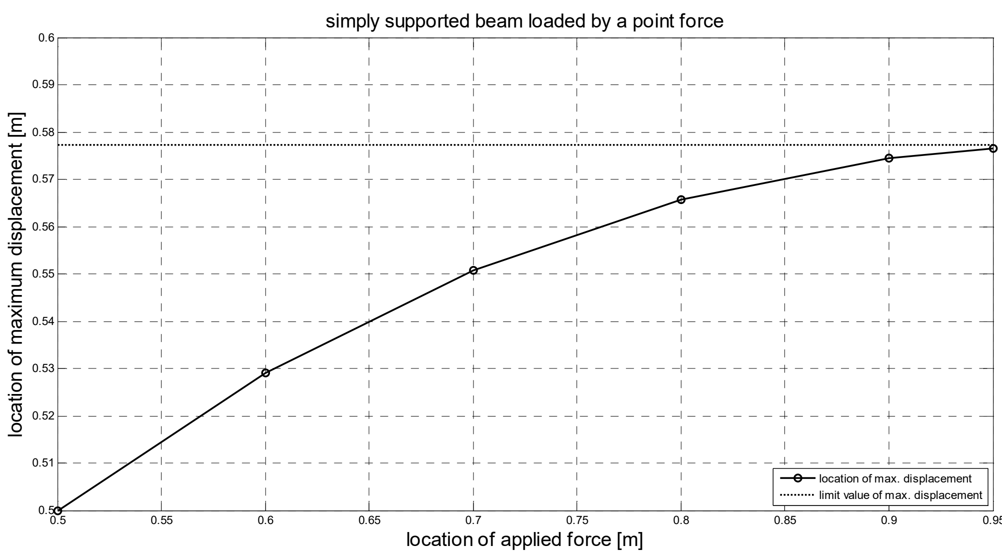


Fig. LM_25 … Location of the maximum deflection as a function of the force location

The Table LM_2 shows how the location of the maximum deflection and the value of the maximum deflection depend on the force location. Notice, how the location of the maximum deflection approaches to the theoretical limit value given by Eq. (LM_120).

Force location	location of max. deflection	value of max. deflection
5.0e-001	4.999999999999999e-001	1.904761904761905e-004
6.0e-001	5.291502622129181e-001	1.806166228353427e-004
7.0e-001	5.507570547286101e-001	1.527432898447346e-004
8.0e-001	5.656854249492380e-001	1.103355952662894e-004
9.0e-001	5.744562646538028e-001	5.777388718803958e-005
9.5e-001	5.766281297335397e-001	2.921582523983270e-005

Table LM_2 … Locations of force, locations of max. deflection and the value of max. deflection

See the program **mpp_012e_beam_deflection_slope_single_force_ccc**.

```
% mpp_012e_beam_deflection_slope_single_force_ccc
% deflection and slope for a simply supported beam of the lenhgt l
% loaded by a point force a a distance a from the left support

clear; format long e
l = 1;                  % beam length
b1 = 0.05; h1 = 0.05;   % dimensions of rectangular cross section
Jy = b1*h1^3/12;        % cross-sectional moment
F = 1000;               % force is applied at a distance 'a' from the left support
E = 2.1e11;             % Young modulus

incr = 0.01;            % 'x' variable increment
xrange = [0:incr:l];    % range of 'x' variable
arange = l*[0.5 0.6 0.7 0.8 0.9 0.95];  % range of 'a' variable
% dimensions of arrays
ss = zeros(length(xrange), length(arange));
ww = ss;
xmax = zeros(length(arange),1);

const3 = F/(2*E*Jy);
const4 = F/(6*E*Jy);
ia = 0;                 % distance counter
for a = arange          % 'a' loop
    ia = ia + 1;
    b = l - a;
    const1 = F*b/(6*l*E*Jy);
    ix = 0;
    for x = xrange      % 'x'loop
        const2 = const1*x;
        ix = ix + 1;
        if x <= a           % the first part of the beam
            ss(ix,ia) = const1*(l^2-b^2-3*x^2);     % slope
            ww(ix,ia) = const2*(l^2-b^2-x^2);       % displacement
        else                % the second part of the beam
            ss(ix,ia) = const1*(l^2-b^2-3*x^2) + const3*(x -a)^2;
            ww(ix,ia) = const2*(l^2-b^2-x^2) + const4*(x - a)^3;
        end
        xmax(ia) = sqrt(1/3)*sqrt(l^2 - b^2);  % location of max. dospl.
        wmax(ia) = F*b*((l^2)-b^2)^(3/2)/(9*sqrt(3)*E*Jy*l);
        aa(ia) = a;                            % location of force
    end
end
disp('    force applied at          max. displacment is at    maximum displacement')
disp([aa' xmax wmax'])

ax = [0 l -7e-4 7e-4];                          % axis argument
figure(1)
ia = 0;
for a = arange
    ia = ia + 1;
    xx = xrange; yy1 = ss(:,ia); yy2 = ww(:,ia);
    subplot(2,3,ia)
    txt =['force is at ' num2str(aa(ia)) '*length'];
    plot(xx,yy1,'k:', xx,yy2,'k-', xmax(ia),0,'o', a,0,'d', ...
        'linewidth',2, 'MarkerSize',10 )
    grid; axis(ax);  title(txt, 'fontsize', 16);
    if ia >= 4, xlabel('length', 'fontsize', 16); end
    if (ia == 1) | (ia == 4), ylabel('displacemnt and slope', 'fontsize', 16); end
    if ia == 6,
        legend('slope [rad]', 'displacement [m]', 'location of max. displ.', 'location of force',
3)
    end
end

figure(2)
isqrt3 = 1/sqrt(3);
xx = [0.5 0.95]; yy = [isqrt3 isqrt3];
plot(aa, xmax,'k-o', xx,yy,'k:', 'linewidth',2); axis([0.5 0.95 0.5 0.6]);
```

```
grid;
legend('location of max. displacement', 'limit value of max. displacement', 4)
xlabel('location of applied force [m]', 'fontsize',16);
ylabel('location of maximum displacement [m]', 'fontsize',16)
title('simply supported beam loaded by a point force', 'fontsize',16 )
```

Example – cantilever beam subjected to a uniformly distributed load of constant intensity

Given: dimensions, load q $[\mathrm{N/m}]$. See Fig. LM_26.
Determine: The deflection and the slope as functions of the longitudinal coordinate.

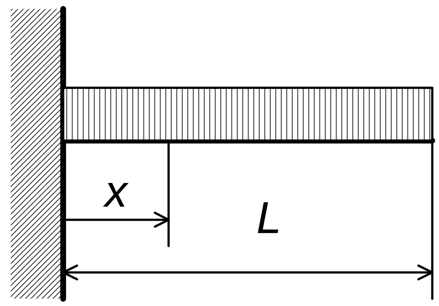


Fig. LM_26 … continuously loaded cantilever beam

The bending moment as a function of the x-coordinate is

$$M(x) = M_A - R_A x + qx\frac{x}{2} = gl\frac{l}{2} - glx + qx\frac{x}{2}. \qquad \text{(LM\_123)}$$

The differential equation of the deflection curve was derived in the form

$$w''(x) = -\frac{M(x)}{EJ_y}. \qquad \text{(LM\_124)}$$

Substituting Eq. (LM_123) into Eq. (LM_124) and after by two consecutive integrations, we get

$$w''(x) = \frac{1}{EJ_y}\left(gl\frac{l}{2} - glx + qx\frac{x}{2}\right), \qquad \text{(LM\_125)}$$

$$w'(x) = \frac{1}{EJ_y}\left(q\frac{l^2}{2}x - ql\frac{x^2}{2} + q\frac{x^3}{6} + C_1\right), \qquad \text{(LM\_126)}$$

$$w(x) = \frac{1}{EJ_y}\left(ql^2\frac{x^2}{4} - ql\frac{x^3}{6} + q\frac{x^4}{24} + C_1 x + C_2\right). \qquad \text{(LM\_127)}$$

The beam is clamped on the left. This means that no deflection and no rotation are permitted at this location. This constraint, expressed mathematically, means that

$$w(0) = 0 \text{ and } w'(0) = 0. \qquad \text{(LM\_128)}$$

Substituting the boundary conditions into Eqs. (LM_126) and (LM_127) we get

$$C_1 = 0 \text{ and } C_2 = 0. \qquad \text{(LM\_129)}$$

The slope of the deflection curve and the deflection as functions of the x-coordinate are

$$\varphi(x) = w'(x) = \frac{q}{6EJ_y}\left(3l^2x - 3lx^2 + x^3\right), \qquad \text{(LM\_130)}$$

$$w(x) = \frac{q}{24EJ_y}\left(6l^2x^2 - 4lx^3 + x^4\right). \qquad \text{(LM\_131)}$$

See Fig. Fig. LM_27 and the program **mpp_013e_cantilever_beam_distributed_loading_c1** This time, the positive deflection was plotted downwards.

The maximum slope and the maximum deflection for a continuously loaded cantilever beam occur at its free end, i.e. for $x = L$.

$$w'_{\max} = w'(L) = \frac{qL^3}{6EJ_y}, \qquad w_{\max} = w(L) = \frac{qL^4}{8EJ_y}. \qquad \text{(LM\_132), (LM\_133)}$$

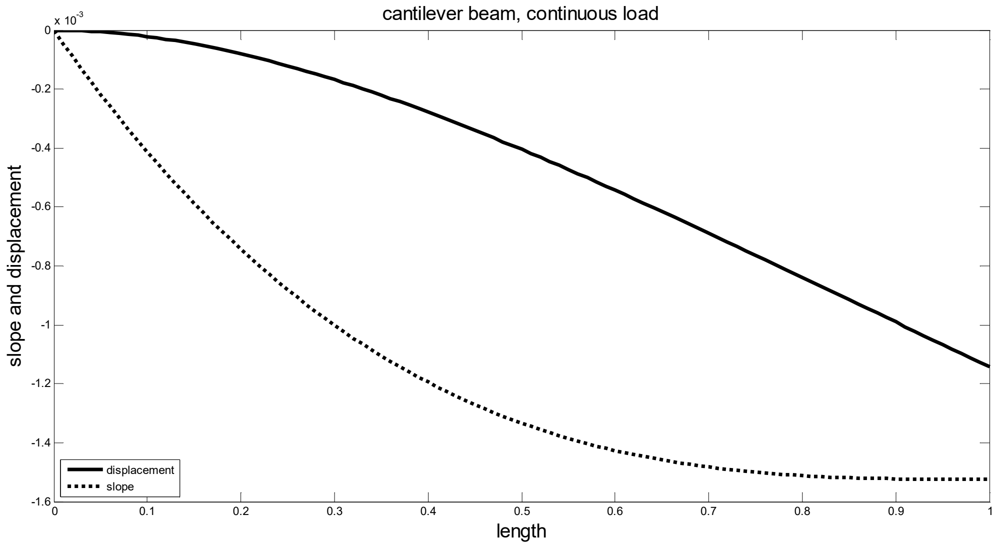


Fig. LM_27 … Slope and deflection for a continuously loaded cantilever beam

Program **mpp_013e_cantilever_beam_ditributed_loading_c1**

```
% mpp_013e_cantilever_beam_ditributed_loading_c1
% deflection and slope of a cantilever beam
% which is continuously loaded

clear; format long e
l = 1;                    % beam length
b1 = 0.05; h1 = 0.05;     % dimensions of rectangular cross section
Jy = b1*h1^3/12;          % cross-sectional moment
q = 1000;                 % loading [N/m]
E = 2.1e11;               % Young modulus

incr = 0.01;              % 'x' variable increment
```

```
xrange = [0:incr:l];    % range of 'x' variable

const1 = q/ (6*E*Jy);
const2 = q/(24*E*Jy);

ix = 0;
for x = xrange
    ix = ix + 1;
    s(ix) = -const1*(3*l^2*x - 3*l*x^2 + x^3);
    w(ix) = -const2*(6*l^2*x^2 - 4*l*x^3 +  x^4);
end

figure(1)
plot(xrange,w, 'k-', xrange,s, 'k:', 'linewidth', 3)
% axis([0 1 -1.6e-3 0.05e-3])
title('cantilever beam, continuous load', 'fontsize', 16)
xlabel('length', 'fontsize', 16)
ylabel('slope and displacement', 'fontsize', 16)
legend('displacement', 'slope', 3)

smax = q*l^3/(6*E*Jy)       % max. slope at the end
wmax = q*l^4/(8*E*Jy)       % max. displacement at the end


Maximum slope in [rad] and maximum deflection in [m] are
smax = 1.523809523809524e-003, wmax = 1.142857142857143e-003.
```

Example – cantilever beam loaded at the free end by a single force

Given: Cantilever beam, dimensions, force F Fig. LM_28.
Determine: Distribution of slope and displacement along the beam.

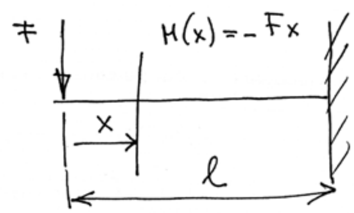


Fig. LM_28 … Cantilever beam force

Bending moment

$$M(x) = -Fx. \qquad \text{(LM\_134)}$$

Differential equation and its integration

$$w''(x) = -\frac{M(x)}{EJ_y} = \frac{F}{EJ_y}x, \qquad \text{(LM\_135)}$$

$$w'(x) = \frac{F}{2EJ_y}x^2 + C_1, \qquad \text{(LM\_135)}$$

$$w(x) = \frac{F}{6EJ_y}x^3 + C_1x + C_2. \qquad \text{(LM\_136)}$$

Boundary conditions for the free end of the beam

$$w'(0) = 0 \quad \Rightarrow \quad C_1 = -\frac{Fl^2}{2EJ_y}, \qquad \text{(LM\_137)}$$

$$w(x) = 0 \quad \Rightarrow \quad C_2 = \frac{Fl^3}{3EJ_y}. \qquad \text{(LM\_138)}$$

Finally, the slope and displacement, plotted in Fig. LM_29, are

$$w'(x) = \frac{F}{2EJ_y}\left(x^2 - l^2\right), \qquad \text{(LM\_139)}$$

$$w(x) = \frac{F}{6EJ_y}\left(x^3 - 3l^2 x + 2l^3\right). \qquad \text{(LM\_140)}$$

The maximum slope and the maximum displacement occur at the free end, i.e. for $x = 0$. So,

$$w'_{\max} = \frac{Fl^2}{2EJ_y} \quad \text{and} \quad w_{\max} = \frac{Fl^3}{3EJ_y} \qquad \text{(LM\_141), (LM\_142)}$$

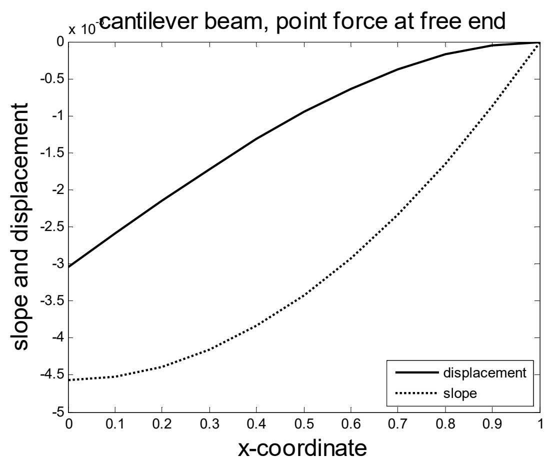


Fig. LM_29 … Cantilever beam point force at the free end.

The slope and displacement distributions are evaluated and plotted by the program **mpp_014e_cantilever_beam_single_force**

```
% mpp_014e_cantilever_beam_single_force
% deflection and slope of a cantilever beam
% which is loaded by a point force at the free end

clear; format long e
l = 1;                  % beam length
b1 = 0.05; h1 = 0.05;   % dimensions of rectangular cross section
Jy = b1*h1^3/12;        % cross-sectional moment
F = 1000;               % loading [N]
E = 2.1e11;             % Young modulus
```

```
incr = 0.1;             % 'x' variable increment
xrange = [0:incr:l];     % range of 'x' variable

const1 = F/(2*E*Jy);
const2 = F/(6*E*Jy);

ix = 0;
for x = xrange
    ix = ix + 1;
    s(ix) = const1*(x^2 - l^2);
    w(ix) = -const2*(x^3 - 3*l^2*x + 2*l^3);
end

figure(1)
plot(xrange,w, 'k-', xrange,s, 'k:', 'linewidth', 2)
title('cantilever beam, point force at free end', 'fontsize', 16)
xlabel('x-coordinate', 'fontsize', 16)
ylabel('slope and displacement', 'fontsize', 16)
legend('displacement', 'slope', 4)
```

Example – statically indeterminate beam subjected to uniformly distributed load

Given: Dimensions, distributed load $q\,[N/m]$.
See Fig. LM_30.
Determine: Distributions of slope and displacements along the beam

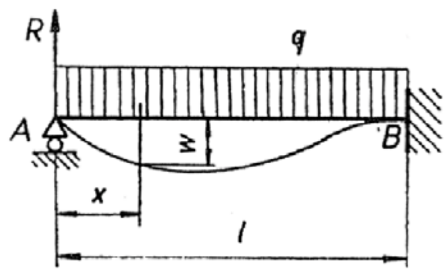


Fig. LM_30 … Beam indeterminate distributed load

Since the problem is statically indeterminate, the equilibrium equations do not suffice for the determination of reactions. In this case, the role of the additional deformation condition will be played by the equation of the deflection curve.

At the support A, there is a vertical reaction R. So far, it is unknown. Then, the bending moment at the distance x is

$$M(x) = Rx - qx\frac{x}{2}. \qquad \text{(LM\_143)}$$

Substituting into $w''(x) = -\dfrac{M(x)}{EJ_y}$ we get

$$w''(x) = -\frac{1}{EJ_y}\left(Rx - \frac{1}{2}qx^2\right).$$

Subsequent integrations give

$$w'(x) = -\frac{1}{EJ_y}\left(\frac{1}{2}Rx^2 - \frac{1}{6}qx^3 + C_1\right), \qquad \text{(LM\_144)}$$

$$w(x) = -\frac{1}{EJ_y}\left(\frac{1}{6}Rx^3 - \frac{1}{24}qx^4 + C_1x + C_2\right). \qquad \text{(LM\_145)}$$

From the first boundary condition, i.e. $w(0) = 0$, follows that $C_2 = 0$.

The remaining two boundary conditions, i.e. $w'(l) = 0$ a $w(l) = 0,$ provide two equations

$$\frac{1}{2}Rl^2 - \frac{1}{6}ql^3 + C_1 = 0, \qquad \text{(LM\_146)}$$

$$\frac{1}{6}Rl^2 - \frac{1}{6}ql^2 + C_1 = 0. \qquad \text{(LM\_147)}$$

Solving them we get

$$R = \frac{3}{8}ql, \quad C_1 = -\frac{ql^3}{48}. \qquad \text{(LM\_148)}$$

Substituting the expression for R into Eq. (LM_143) we get the bending moment in the form

$$M(x) = \frac{1}{8}qx(3l - 4x), \qquad \text{(LM\_149)}$$

which could be substituted into Eq. (LM_145) for the vertical displacement. So,

$$w(x) = \frac{qx}{48EJ_y}\left(l^3 - 3lx^2 + 2x^3\right). \qquad \text{(LM\_150)}$$

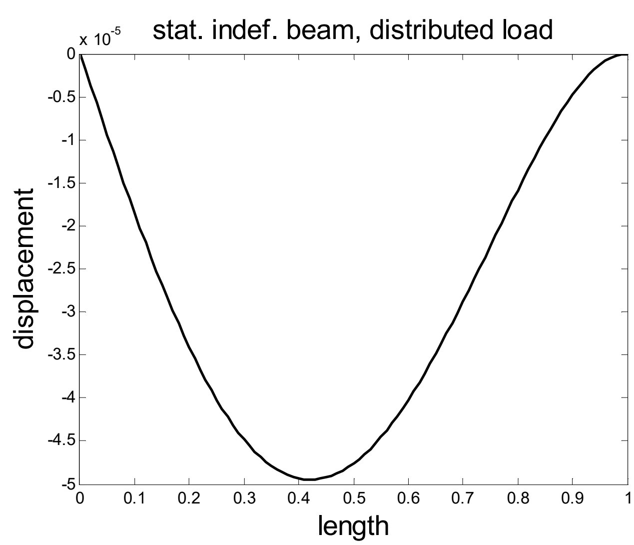


Fig. LM_31 … indeterminate beam distributed loading

The displacement function of the statically indeterminate beam subjected to uniformly distributed load is evaluated by the program **mpp_015e_indeter_beam_distributed_loading** and plotted in Fig. LM_31.

```
% mpp_015e_indeter_beam_distributed_loading
% deflection and slope of a statically indefinite beam
% contnuous soad

clear; format long e
l = 1;                  % beam length
b1 = 0.05; h1 = 0.05;   % dimensions of rectangular cross section
Jy = b1*h1^3/12;        % cross-sectional moment
q = 1000;               % loading [N]
E = 2.1e11;             % Young modulus

incr = 0.01;            % 'x' variable increment
xrange = [0:incr:l];    % range of 'x' variable

const = q/(48*E*Jy);

ix = 0;
for x = xrange
    ix = ix + 1;
    w(ix) = -const*x*(l^3 - 3*l*x^2 + 2*x^3);
end

figure(1)
plot(xrange,w, 'k-', xrange,s, 'k..' 'linewidth', 2)
title('stat. indef. beam, continuous load', 'fontsize', 16)
xlabel('length', 'fontsize', 16)
ylabel('displacement', 'fontsize', 16)
```

Example – deformation of beams with variable cross sections

There is nothing new when solving this type of task – the analysis, however, is lengthier and has to be carried out by parts.

Given: Dimensions, force F, see Fig. LM_32.
Determine: Distribution of the slope and the displacement as a function of the beam length.

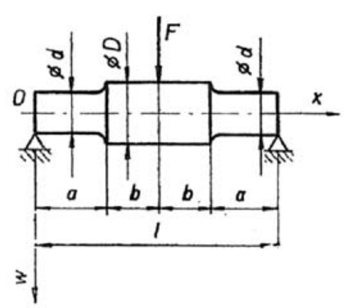


Fig. Fig. LM_32 ... Beam variable cross section.

Evaluate the reactions at first. Due to the symmetry of loading both reactions are the same and equal to $F/2$. Due to the geometrical symmetry it suffices to solve only the first part of the beam, i.e. $0 \le x \le l/2$.

In the first part of the beam the bending moment is

$$M(x) = \frac{F}{2}x. \qquad \text{(LM\_151)}$$

There are two different cross sections and thus

$$J_1 = J_y = \frac{\pi}{64} d^4 \qquad \text{for } 0 \le x \le a, \qquad \text{(LM\_152)}$$

$$J_2 = J_y = \frac{\pi}{64} D^4 \qquad \text{for } a \le x \le l/2. \qquad \text{(LM\_153)}$$

In the first interval, i.e. for $0 \le x \le a$, we have

$$w_1''(x) = -\frac{F}{2EJ_1} x, \qquad \text{(LM\_154)}$$
$$w_1'(x) = -\frac{F}{4EJ_1} x^2 + C_1, \qquad \text{(LM\_155)}$$
$$w_1(x) = -\frac{F}{12EJ_1} x^3 + C_1 x + C_2. \qquad \text{(LM\_156)}$$

In the second interval, i.e. for $a \le x \le l/2$, we have

$$w_2''(x) = -\frac{F}{2EJ_2} x, \qquad \text{(LM\_157)}$$
$$w_2'(x) = -\frac{F}{4EJ_2} x^2 + C_3, \qquad \text{(LM\_158)}$$
$$w_2(x) = -\frac{F}{12EJ_2} x^3 + C_3 x + C_4. \qquad \text{(LM\_159)}$$

Boundary conditions

$w_1(0) = 0$	zero displacement in the left support,	(LM_160)
$w_2'(l/2) = 0$	zero slope in the middle of the beam (symmetry),	(LM_161)
$w_1(a) = w_2(a)$	distribution of displacements has to be continuous,	(LM_162)
$w_1'(a) = w_2'(a)$	distribution of slopes has to be continuous.	(LM_163)

From the boundary conditions we get

$$C_1 = \frac{Fa^2}{4E} \frac{J_2 - J_1}{J_1 J_2} + \frac{Fl^2}{16EJ_2}, \qquad \text{(LM\_164)}$$
$$C_2 = 0, \qquad \text{(LM\_165)}$$
$$C_3 = \frac{Fl^3}{16EJ_2}, \qquad \text{(LM\_166)}$$
$$C_4 = \frac{Fa^3}{6E} \frac{J_2 - J_1}{J_1 J_2}. \qquad \text{(LM\_167)}$$

The maximum displacement is in the middle of the beam

$$w_{\max} = w_2\left(\frac{l}{2}\right) = \frac{Fa^3}{6E}\frac{J_2 - J_1}{J_1 J_2} + \frac{Fl^3}{48EJ_2}. \tag{LM\_168}$$

The distributions of the slope and vertical displacement is evaluated by the program **mpp_016e_different_cross_sections_single_force** for the following input data

```
% beam length
l = 1;
% diam. of cross sections
d = 0.05; D = 0.1;
J1 = pi*d^4/64;
J2 = pi*D^4/64;
% loading [N]
F = 1000;
% Young modulus
E = 2.1e11;
% coordinate
a = 0.3;
```

The graphical output is in Fig. LM_33.

The displacement in the location of a sudden change of the cross section is continuous. The derivative of the deflection curve (i.e. the slope) is continuous as well but is not smooth. From it follows that the second derivative of the displacement curve (i.e. bending the moment) makes a sudden jump in the distribution. But the bending moment is proportional to the stress. And the jump in the stress distribution is theoretically impossible to achieve. Thus, the theory is approximate. Heisenberg, however, said that there are no jumps in the Nature, since the Nature is full of jumps.

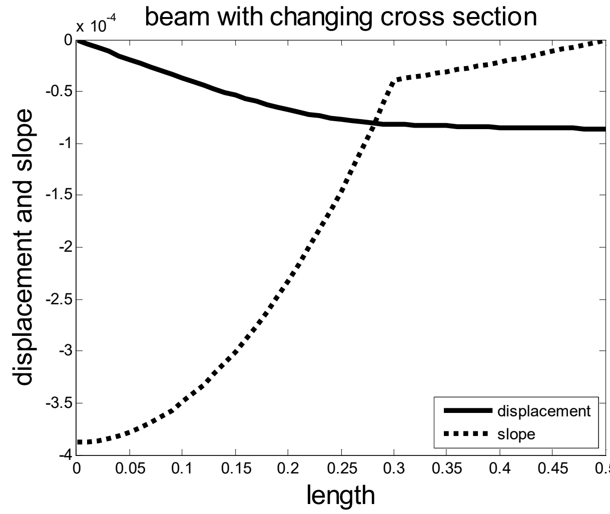


Fig. LM_33... Beam with changing cross section

The program evaluating the task is **mpp_016e_different_cross_sections_single_force**

```
% mpp_016e_different_cross_sections_single_force
% simply supported beam
% different cross sections
% symmetry

clear; format long e
% beam length
l = 1;
% diameters of cross sections
d = 0.05; D = 0.1;
```

```
J1 = pi*d^4/64;
J2 = pi*D^4/64;
% loading [N]
F = 1000;
% Young modulus
E = 2.1e11;
% coordinate
a = 0.3;

b = (l - 2*a)/2;

incr = 0.01;               % 'x' variable increment
xrange = [0:incr:l/2];     % range of 'x' variable

C1 = F*a^2*(J2 - J1)/(4*E*J1*J2) + F*l^2/(16*E*J2);
C2 = 0;
C3 = F*l^2/(16*E*J2);
C4 = F*a^3*(J2 - J1)/(6*E*J1*J2);

const1 = F/(12*E*J1);
const2 = F/(12*E*J2);
const3 = F/(4*E*J1);
const4 = F/(4*E*J2);

ix = 0;
for x = xrange
    ix = ix + 1;
    if x <= a,
       s(ix) = -const3*x^2 + C1;
       w(ix) = -const1*x^3 + C1*x + C2;
    else
       s(ix) = -const4*x^2 + C3;
       w(ix) = -const2*x^3 + C3*x + C4;
    end
end

wmax = F*a^3*(J2 - J1)/(6*E*J1*J2) + F*l^3/(48*E*J2)

figure(1)
plot(xrange,-w, 'k-', xrange,-s, 'k:', 'linewidth', 2)
title('beam with changing cross sections', 'fontsize', 16)
xlabel('length', 'fontsize', 16)
ylabel('displacement and slope', 'fontsize', 16)
legend('displacement', 'slope', 4)
```

Example – the complete differential equation of the deflection curve

From the relations derived and presented so far

$$
\begin{aligned}
&w''(x) = -\frac{1}{EJ_y} M(x), \\
&q(x) = -\frac{\mathrm{d}T(x)}{\mathrm{d}x} = -T'(x), \\
&T(x) = -\frac{\mathrm{d}M(x)}{\mathrm{d}x} = -M'(x), \\
&M''(x) = -q(x),
\end{aligned}
\qquad \text{... (LM\_169)}
$$

the complete differential equation of the deflection curve could be derived in the form

$$\frac{\mathrm{d}^4 w(x)}{\mathrm{d}x^4} = w^{\mathrm{IV}}(x) = -\frac{1}{EJ_y} M''(x) = \frac{1}{EJ_y} q(x). \qquad \text{(LM\_170)}$$

Example – complete differential equation for a beam with distributed loading

Given: simply supported beam, dimensions, uniformly distributed loading q. Fig. LM_34.
Determine: Distributions of the slope and of the deflection along the beam length, and the distributions of the bending moment and of the shear force.

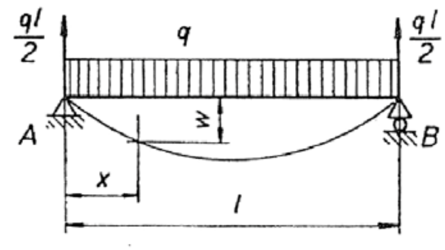


Fig. LM_34 ... Simply supported beam cont load

The task was already solved by another method before. Alternatively, using the above relations (LM_169) and assuming that $q(x) = q = const$, we could write

$$w^{\mathrm{IV}}(x) = \frac{q}{EJ_y}. \qquad \text{(LM\_171)}$$

Integrating we get

$$w'''(x) = \frac{q}{EJ_y} x + C_1, \qquad \text{(LM\_172)}$$

$$w''(x) = \frac{q}{EJ_y} \frac{x^2}{2} + C_1 x + C_2, \qquad \text{(LM\_173)}$$

$$w'(x) = \frac{q}{EJ_y} \frac{x^3}{6} + C_1 \frac{x^2}{2} + C_2 x + C_3, \qquad \text{(LM\_174)}$$

$$w(x) = \frac{q}{EJ_y} \frac{x^4}{24} + C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4. \qquad \text{(LM\_175)}$$

The unknown constants are found from the boundary conditions.

1)
No deflection at the left support … $w(0) = 0$.
2)
No moment at the left support … $M(0) = 0$, and since $w''(x) = M(x)/(EJ_y)$, then $w''(0) = 0$.
3)
No deflection at the right support … $w(l) = 0$.
4)
No moment at the right support … $M(l) = 0$, and since $w''(x) = M(x)/(EJ_y)$, then $w''(l) = 0$.

From the first condition … $C_4 = 0$.

From the second condition ... $C_2 = 0$.

The fourth condition gives ... $C_1 = -\frac{1}{2}\frac{ql}{EJ_y}$.

The third condition gives ... $C_3 = \frac{1}{24}\frac{ql^3}{EJ_y}$.

Substituting the constants into the equations for the slope and the deflection we get

$$w(x) = \frac{qx}{24EJ_y}\left(l^3 - 2lx^2 + x^3\right), \tag{LM\_176}$$

$$w'(x) = \frac{q}{24EJ_y}\left(l^3 - 6lx^2 + 4x^2\right). \tag{LM\_177}$$

The second derivative of the deflection curve is proportional to the bending moment

$$w''(x) = -\frac{1}{EJ_y}M(x). \tag{LM\_178}$$

So,

$$M(x) = -w''(x)EJ_y = q\left(\frac{l}{2}x - \frac{x^2}{2}\right) = \frac{q}{2}\left(lx - x^2\right). \tag{LM\_179}$$

And similarly for the shear force that corresponds to the third derivative

$$T(x) = -w'''(x)EJ_y = q\left(\frac{l}{2} - x\right). \tag{LM\_180}$$

Example – statically indeterminate beam with uniformly distributed loading

Given: Statically indeterminate beam, length $l = 1\ \mathrm{m}$, uniformly distributed loading $q = 40\ \mathrm{kN/m}$, surface area $S = 12\ \mathrm{cm}^2 = 12\times10^{-4}\mathrm{m}^2 = 0.00012\ \mathrm{m}^2$, $W_o = 39.7\ \mathrm{cm}^3 = 397\times10^{-5}\ \mathrm{m}^3$, Young's modulus $E = 2.1\times10^{11}\ \mathrm{Pa}$. See Fig. LM_35.

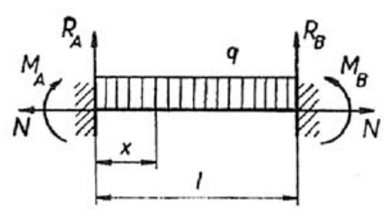


Fig. LM_35 ... Stat indeterminate beam distributed loading

Determine: Distributions of the slope and of the deflection along the beam length and the distributions of the bending moment and of the shear force.

Due to static indeterminacy, the normal (axial) force N arises. Due to symmetry, the vertical components of reactions are identical, i.e.

$$R_{\mathrm{A}} = R_{\mathrm{B}} = \frac{ql}{2}. \tag{LM\_181}$$

Even if the reaction moment M_{A} is unknown so far, the distribution of the bending moment is

$$M(x) = M_{\mathrm{A}} + R_{\mathrm{A}} x - \frac{1}{2} q x^2 = M_{\mathrm{A}} + \frac{ql}{2} x - \frac{1}{2} q x^2 = M_{\mathrm{A}} + \frac{q}{2}\left(lx - x^2\right). \tag{LM\_182}$$

Now, the standard procedure is applied, i.e.

$$w''(x) = -\frac{1}{EJ_y} M(x) = -\frac{1}{EJ_y}\left(M_{\mathrm{A}} + \frac{q}{2}\left(lx - x^2\right)\right), \tag{LM\_183}$$

$$w'(x) = -\frac{1}{EJ_y}\left(M_{\mathrm{A}} x + \frac{ql}{4} x^2 - \frac{q}{6} x^3\right) + C_1, \tag{LM\_184}$$

$$w(x) = -\frac{1}{EJ_y}\left(\frac{1}{2} M_{\mathrm{A}} x^2 + \frac{ql}{12} x^4\right) + C_1 x + C_2. \tag{LM\_185}$$

Boundary conditions.

At the left support (clamping) the slope of the deflection curve has to be equal to zero

$$w'(0) = 0 \Rightarrow C_1 = 0.$$

At the left support (clamping) the displacement of the deflection curve has to be equal to zero

$$w(0) = 0 \Rightarrow C_2 = 0.$$

At the right support (clamping) the slope of the deflection curve has to be equal to zero

$$w'(l) = 0 \Rightarrow \quad M_{\mathrm{A}} = -\frac{ql^2}{12}.$$

Thus, the equation of the deflection curve is

$$w(x) = \frac{q}{24EJ_y} x^2 \left(l - x\right)^2. \tag{LM\_186}$$

Then, the distribution of the bending moment is

$$M(x) = M_A + \frac{q}{2}\left(lx - x^2\right) = -\frac{ql^2}{12} + \frac{q}{2}(lx - x^2) = \frac{q}{12}\left(6lx - l^2 - 6x^2\right). \qquad \text{(LM\_187a)}$$

Its maximum value occurs at the location of supports, i.e. for $x = 0$ and for $x = l$. In absolute value we get

$$\left|M_{\max}\right| = \frac{ql^2}{12} = 3333 \text{ Nm}. \qquad \text{(LM\_187b)}$$

The corresponding stress is

$$\sigma_{\max} = \frac{\left|M_{\max}\right|}{W_o} = 8.39 \times 10^5 \text{ Pa}. \qquad \text{(LM\_188)}$$

The distribution of the displacements along the beam length is evaluated by the program **mpp_017e_beam_clamped_at_both_sides** and plotted in Fig. LM_36.

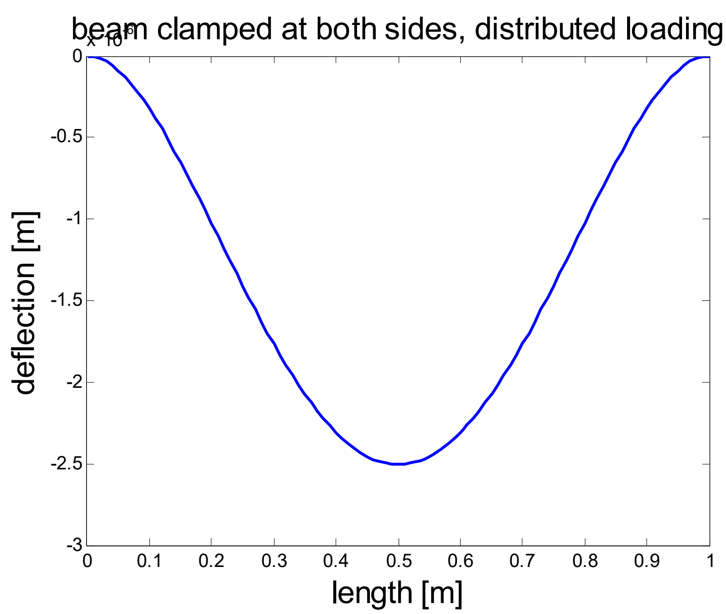


Fig. LM_36 ... Stat indeterminate beam distributed loading results

Program **mpp_017e_beam_clamped_at_both_sides**

```
% mpp_017e_beam_clamped_at_both_sides
% uniform distributed loading

clear; format long e
% beam length
l = 1;
S = 12e-4;
Jy = 198e-6;
W0 = 397e-5;
q = 40000;
```

```
E = 2.1e11;

incr = 0.01;
xrange = 0:incr:l;
const = q/(24*E*Jy);

ix = 0;
for x = xrange
    ix = ix + 1;
    w(ix) = const*x^2*(l - x)^2;
end

Mmax = q*l^2/12
Smax = Mmax/W0

figure(1)
plot(xrange,-w, 'linewidth', 2)
title('beam clamped at both sides, distributed loading', 'fontsize', 16)
xlabel('length [m]', 'fontsize', 16)
ylabel('deflection [m]', 'fontsize', 16)
```

9.2.12. Strain energy in a beam subjected to pure bending

In Fig. LM_37 there is depicted a part of the beam subjected to pure bending. We have derived that the axial stress due to the bending is

$$\sigma_x = \frac{M(x)}{J_y} z .$$

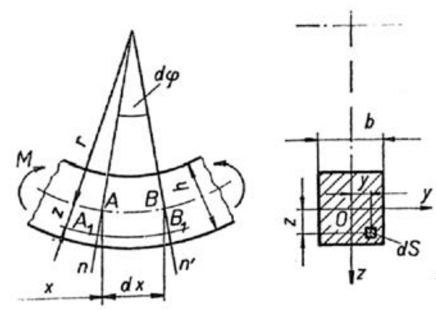


Fig. LM_37 ... Beam defo 1

The pure bending means that the axial stress is of uni-axial nature, and that the influence of shear forces are non-existent or neglected. So, the strain energy of pure bending is analogous to the strain energy in tension – compression as it is reminded in Fig. LM_38. So, the elementary strain energy, contained in an element of a beam between two infinitesimally close slices, depicted in Fig. LM_37, is

$$\mathrm{d}U = \left(\frac{\sigma_x^2}{2E}\mathrm{d}S\right)\mathrm{d}x = \left[\left(\frac{M^2(x)}{2EJ_y^2}\right)\int_S z^2\,\mathrm{d}S\right]\mathrm{d}x =$$

$$= \frac{M^2(x)}{2EJ_y}\,\mathrm{d}x\,. \qquad \text{(LM\_189)}$$

To explain the analogy of bending with the tension it should be reminded that for a bar of the length l we write

$$\sigma = E\varepsilon \quad \frac{P}{S} = E\frac{\Delta l}{l}; \quad P = \frac{ES}{l}\Delta l\,.$$

... (LM_190)

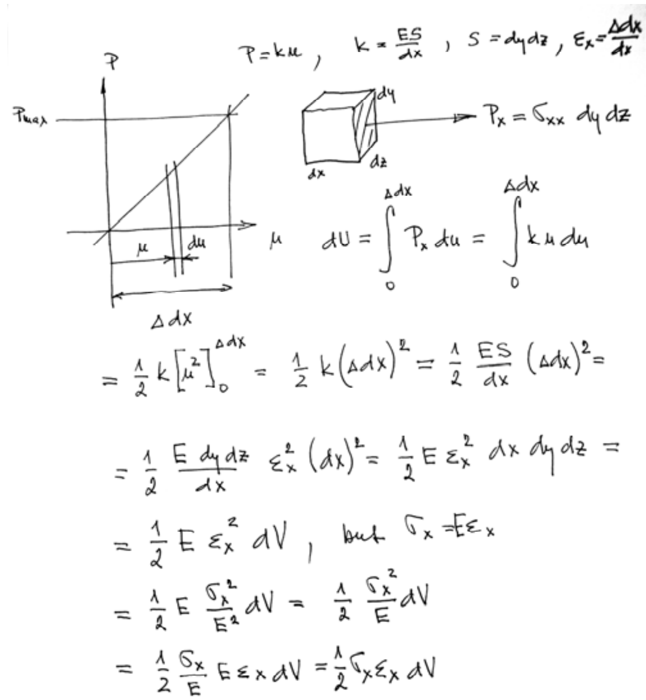


Fig. LM_38 ... 1D strain energy

And similarly for a beam element of the length $\mathrm{d}x$

$$P = \frac{ES}{\mathrm{d}x}\Delta\mathrm{d}x; \quad P = k\,\Delta\mathrm{d}x; \quad k = \frac{ES}{\mathrm{d}x}. \qquad \text{(LM\_191)}$$

The strain energy contained in the whole beam is obtained by the integrating Eq. (LM_189). If $J_y = const$, then

$$U = \frac{1}{2EJ_y}\int_0^l M^2(x)\,\mathrm{d}x\,. \qquad \text{(LM\_192)}$$

Survey of elementary types of the simply supported beam and the corresponding distributions of shear forces and bending moments $T(x)$ and $M(x)$ is depicted in Table LM_3.

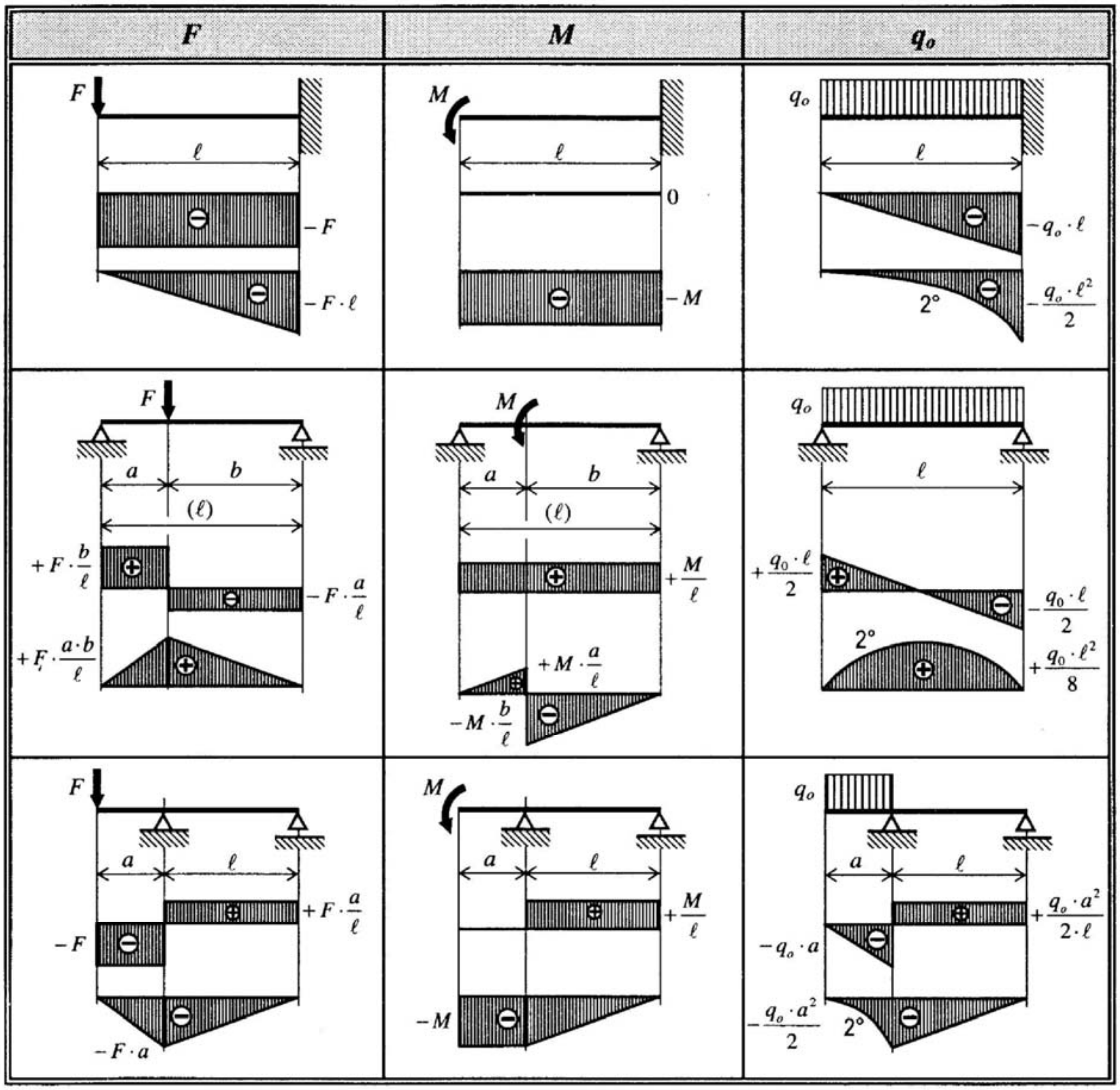


Table LM_3 ... Beam_survey_of_T_and_M

Lot of examples could be found in [21].

9.3. Torsion

9.3.1. Introduction

By torsion we understand the twisting of a body when it is loaded by moments tending to produce the rotation about the longitudinal axis of that body. In the text, we limit our attention to slender prismatic circular bars (rods). In this subsection, the figures are numbered from 50, the equations from 200, and the tables from 10.

Consider a circular prismatic bar, depicted in Fig. LM_50, which is clamped at its left end and subjected to the twisting couple $M = Pp$. Within the linear theory of elasticity, the right cross-sectional surface rotates with respect to the left surface by a small angle φ known as the *angle of twist*. The value of the angle of the twist varies linearly between the left and right surfaces from zero to its maximum. The radial rays stay straight. The initially straight line cd will become a helical curve cd'. Every cross section is subjected to the same torque – it remains planar and does not change its radius. This way, the state of the pure shear occur within the twisted bar.

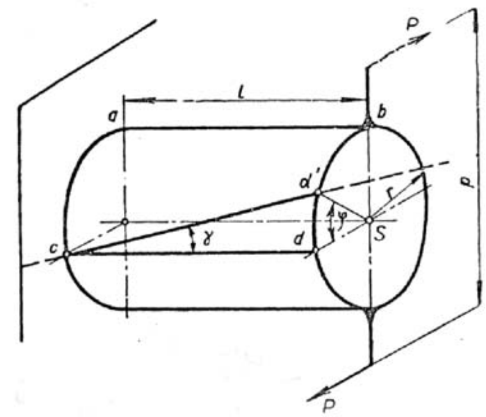


Fig. LM_50 ... Torsion 1

9.3.2. Deformation, strain and stress

Let's analyze how the element of the bar during the twist deformation is deformed. In Fig. LM_51 there is an elementary ring element with radius ρ (this quantity varies from zero to the outer bar radius R) and of the thickness $\mathrm{d}\rho$ and of the length $\mathrm{d}x$.

The relative angular displacement of two layers, displaced by the distance $\mathrm{d}x$, is $\mathrm{d}\varphi$.

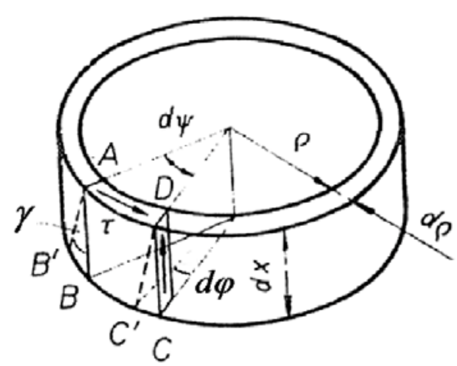


Fig. LM_51 ...Torsion 2

Considering a sector element $\mathrm{d}\psi$ before and after the deformation (the latter is plotted by dashed lines) we can notice that the arc BC is displaced – with respect to the arc AD – by the distance

$$BB' \equiv CC' \equiv \rho \mathrm{d}\varphi . \qquad \text{(LM\_200)}$$

Now, a new quantity, i.e. the *shear strain*, defined as an angle $\gamma \approx BAB'$, is introduced. Evidently,

$$\rho \mathrm{d}\varphi = \gamma \mathrm{d}x \quad \Rightarrow \quad \gamma = \rho \frac{\mathrm{d}\varphi}{\mathrm{d}x}. \tag{LM\_201}$$

Due to this deformation, the shear stress in the analyzed element arises. Since we are living in a linear world, the shear stress τ is proportional to the shear strain γ. The proportionality constant is denoted G and is called the *modulus of elasticity in shear*. So,

$$\tau = G\gamma = G\rho \frac{\mathrm{d}\varphi}{\mathrm{d}x}. \tag{LM\_202}$$

This relation is analogous to the relation derived for the uni-axial stress, i.e. $\sigma = E\varepsilon$.

The quantity $\frac{\mathrm{d}\varphi}{\mathrm{d}x}$ corresponds to the relative angular displacement of two cross-section slices, displaced by the unit of length – it is denoted $\vartheta = \frac{d\varphi}{dx}$ and called the a*ngle of twist per unit length* or the *rate of twist*.

So, the previous equation could be rewritten into the form

$$\tau = G\vartheta\rho. \tag{LM\_203}$$

Evidently, the shear stress is proportional to the distance ρ of the element from the axis. The maximum shear stress occurs at the surface of the bar, say R.

9.3.3. How the applied torque (moment) is related to the shear stress in the bar

In Fig. LM_52 there is depicted a hollow cylinder with inner and outer radii r and R respectively. The elementary surface $\mathrm{d}S$, displaced by the distance ρ from the axis, is loaded by the force $\mathrm{d}F = \tau \mathrm{d}S$.

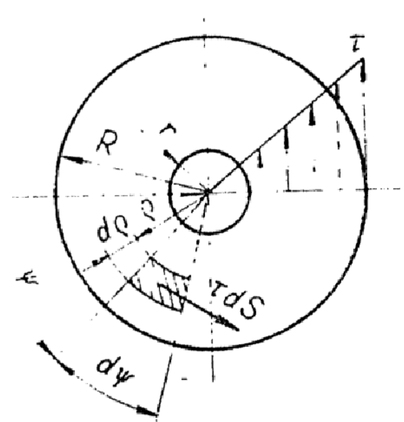


Fig. LM_52 ... Torsion 3

The integral sum of these inner forces has to be in equilibrium with the outer moment M_{k}, so

$$M_{\mathrm{k}} = \int \rho \tau \mathrm{d}S = G\vartheta \int \rho^2 \mathrm{d}S = G\vartheta J_{\mathrm{p}} = GJ_{\mathrm{p}} \frac{\mathrm{d}\varphi}{\mathrm{d}x}, \qquad \text{(LM\_204)}$$

where $\vartheta = \dfrac{\mathrm{d}\varphi}{\mathrm{d}x}$ is the *rate of twist* and

$J_{\mathrm{p}} = \int \rho^2 \, \mathrm{d}S$ is the polar moment of the cross-sectional area.

So,

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = \frac{1}{GJ_{\mathrm{p}}} M_{\mathrm{k}} \Rightarrow \mathrm{d}\phi = \frac{1}{GJ_{\mathrm{p}}} M_{\mathrm{k}} \, \mathrm{d}x \,. \qquad \text{(LM\_205)}$$

Integrating the previous formula with respect to the length, we get the total angle of twist for a bar of the length l loaded by the moment M_{k} in the form

$$\phi = M_{\mathrm{k}} \frac{l}{GJ_{\mathrm{p}}} \,. \qquad \text{(LM\_206)}$$

The variable $\dfrac{l}{GJ_{\mathrm{p}}}$ is called the *torsion flexibility*.

Its inverse value, i.e. $\dfrac{GJ_{\mathrm{p}}}{l}$, is the *torsion stiffness*.

9.3.4. Analogy of relations for tension, bending and torsion

1D stress	bending	torsion
$\sigma = E\varepsilon$	$\sigma = E\varepsilon$	$\tau = G\gamma$ ... Hooke's law
$\sigma = \frac{F}{S} = E\varepsilon = E\frac{\Delta l}{l}$	$\sigma = \frac{M_{\mathrm{o}}}{W_{\mathrm{o}}}$	$\tau = \frac{M_{\mathrm{k}}}{W_{\mathrm{k}}}$ … stress
where M_{o} and M_{k} are bending and torsion moments respectively		
S	$W_{\mathrm{o}} = \frac{J_y}{z_{\max}}$	$W_{\mathrm{k}} = \frac{J_{\mathrm{p}}}{r_{\max}}$
	these relations are valid for circular cross sections only	
area	W_{o} and W_{k} are section modules in bending and torsion	
longitudinal strain	curvature	rate of twist
$\varepsilon = \frac{\Delta l}{l} = \frac{F}{ES}$	$\frac{1}{\rho} = \frac{M_{\mathrm{o}}}{EJ_y}$	$\vartheta = \frac{\mathrm{d}\varphi}{\mathrm{d}x} = \frac{M_{\mathrm{k}}}{GJ_{\mathrm{p}}}$
stiffness		
$F = \underbrace{\frac{ES}{l}}_{\text{longitud. stiffness}} \underbrace{\Delta l}_{\text{elongation}}$	$M_{\mathrm{o}} = \underbrace{EJ_y}_{\text{bending stiffness}} \underbrace{\frac{1}{\rho}}_{\text{curvature}}$	$M_{\mathrm{k}} = \underbrace{\frac{GJ_{\mathrm{p}}}{l}}_{\text{torsional stiffness}} \underbrace{\varphi}_{\text{twist}}$
strength theories		
$\sigma_{\max} = E\varepsilon_{\max} < \sigma_{\mathrm{Dt}}$	$\sigma_{\max} = \frac{M_{\mathrm{o}}}{W_{\mathrm{o}}} < \sigma_{\mathrm{Dt}}$	$\tau_{\max} = \frac{M_{\mathrm{k}}}{W_{\mathrm{k}}} < \tau_{\mathrm{D}}$
where σ_{Dt} is the allowable stress in tension and τ_{D} is the allowable stress in torsion.		
strain energy		
$U = \frac{F^2 l}{2ES}$	$U = \int_0^l \frac{M_o^2(x)}{2EJ_y}\mathrm{d}x$	$U = \frac{M_{\mathrm{k}}^2 l}{2GJ_{\mathrm{p}}}$
constant force	variable moment	constant moment

Table LM_10 ... Analogies

Lot of examples could be found in [21].

9.4. Buckling

9.4.1. Introduction

In engineering, the term *buckling* is related to a loss of stability. The machine part being loaded might loose its geometrical integrity even if other conditions for its save conduct are satisfied. In this chapter we will devote our attention to the axial loading of long slender structural members. In this subsection, the figures are numbered from 200, the equations from 300, and, the tables from 20.

9.4.2. Stability

Generally, there are three types of equilibrium in statics.

They could be classified according to the amount of mechanical work (energy) needed to displace the body from its immediate position. See Fig. LM_200. We distinguish three cases of stability

- stable – a positive energy is required,
- indifferent – no energy is needed,
- unstable – a small mechanical pulse is required, then the body produces mechanical energy.

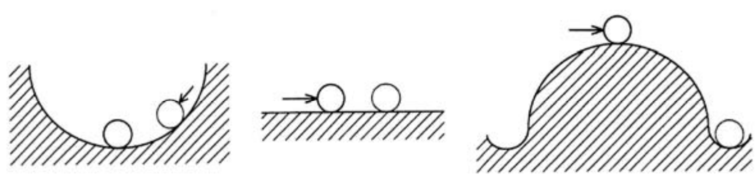


Fig. LM_200 ... Buckling stable unstable

In mechanics of deformable bodies, any deformation could be considered as the possible displacement. In Fig. LM_201 there is depicted a ring loaded by two forces, maintaining, after the deformation, their directions.

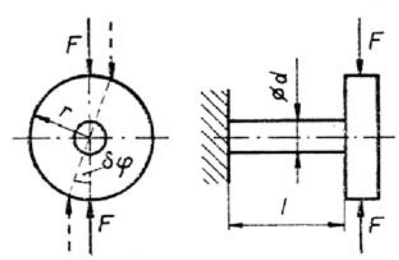


Fig. LM_201 ... Buckling 2

Due to the twist deformation $\delta\varphi$, these two forces exert the mechanical work

$$\delta^2 A = 2Fr(1-\cos\delta\varphi) = 2Fr(1-(1-\frac{(\delta\varphi)^2}{2}+\cdots)) \cong Fr(\delta\varphi)^2 .$$

... (LM_300)

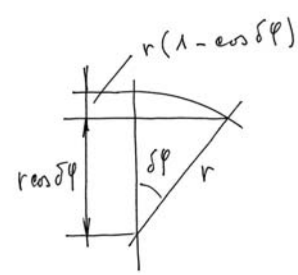


Fig. LM_202 ... Buckling 3

where $r = d/2$. See Fig. LM_201 and Fig. LM_202. In the previous equation the cosine function is approximated by the Taylor's series expansion.

The strain energy increases by

$$\delta^2 U = \frac{1}{2}\delta M_{\mathrm{k}}\,\delta\varphi = \frac{1}{2}GJ_{\mathrm{p}}(\delta\varphi)^2 . \qquad \text{(LM\_301)}$$

The total potential energy is defined as

$$W = U + V , \qquad \text{(LM\_302)}$$

where V – called the potential – was defined as the negative energy due to external forces.

Generally, the equilibrium condition could be alternatively formulated as

$$\delta W = \delta U + \delta V = \delta U - \delta A . \qquad \text{(LM\_303)}$$

It was proved that the stable equilibrium occurs if

$$\delta^2 W = \delta^2 U + \delta^2 V = \delta^2 U - \delta^2 A > 0 . \qquad \text{(LM\_304)}$$

In our example, the condition of stable equilibrium requires $\delta^2 A < \delta^2 U$, so $Fr < \frac{1}{2}GJ_{\mathrm{p}}$ and finally

$$F < \frac{GJ_{\mathrm{p}}}{2\mathrm{r}} . \qquad \text{(LM\_305)}$$

Let's define the critical force by $F_{\mathrm{crit}} = \frac{GJ_{\mathrm{p}}}{2\mathrm{r}}$. If the loading force reaches the critical force, the state of equilibrium becomes indifferent. If the loading force overcomes the critical force, then the state of equilibrium becomes unstable. It means that any – howsoever small – accidentally evoked torsional displacement will be increased without bounds and the loaded bar will lose its stability.

9.4.3. Buckling of slender bars

A straight slender prismatic cylindrical bar[2], depicted in Fig. LM_203, is loaded by two opposite compression forces. The bar, being bended, is on the right. The deflection curve is described by $w = w(x)$. The bending moment, in the cross-section displaced by x, is

$$M = M(x) = F\,w(x)\,. \qquad \text{(LM\_306)}$$

We know that

$$w''(x) = -\frac{M(x)}{EJ_y}\,. \qquad \text{(LM\_307)}$$

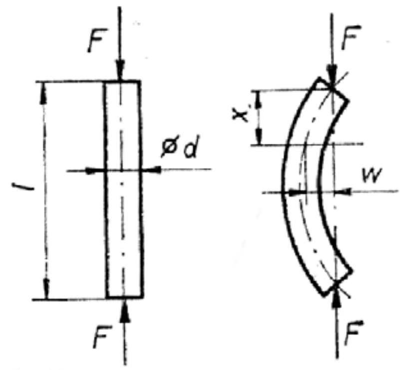


Fig. LM_203 ... Buckling 4

Comparing the last two equations we get

$$w''(x) = -\frac{F\,w(x)}{EJ_y}\,. \qquad \text{(LM\_308)}$$

Introducing a new variable $p^2 = \dfrac{F}{EJ_y}$, the previous equation could be rewritten as

$$w''(x) + p^2 w(x) = 0\,. \qquad \text{(LM\_309)}$$

Then, its solution could be assumed in the form

$$w(x) = A\sin px + B\cos px\,, \qquad \text{(LM\_310)}$$

where A, B are constants of integration. The boundary conditions are

$$w(0) = 0,\; w(l) = 0\,. \qquad \text{(LM\_311)}$$

The former gives $B = 0$, while from the latter we get

$$A\sin pl = 0\,. \qquad \text{(LM\_312)}$$

[2] This kind of machine detail is also called rod or strut.

The above differential condition could be satisfied under two conditions:

either $A = 0$, leading to a trivial solution – the bar does not bend. This is the case which does not interest us,

or the argument of the sine function has to reach the values $pl = 0, \pi, 2\pi, \cdots, k\pi$, where k is an integer.

So, the above differential equation has a non-trivial solution – equilibrium in the state of bending. This might happen if $pl = k\pi$, kde $k = 1, 2, \cdots$, thus for $p = k\pi / l$. The value $k = 0$ was excluded since it leads to the trivial solution again.

So, the solution of the differential equation has the form

$$w(x) = A \sin \frac{k\pi}{l} x . \qquad \text{(LM\_313)}$$

The smallest possible force – the critical force, say F_{crit} – for which this situation could occur, is obtained for $k = 1$. In such a case

$p = (1 \times \pi)/l$ and also $p^2 = \dfrac{F_{\text{crit}}}{EJ_y}$. From it follows

$$F_{\text{crit}} = \frac{\pi^2 E J_y}{l^2} . \qquad \text{(LM\_314)}$$

This force, known as the Euler's critical force, specifies the limit load that leads to the loss of the buckling stability. The corresponding critical stress is $\sigma_{\text{crit}} = -F_{\text{crit}} / S$, where S is the cross-sectional area of the beam. The validity of Hooke's law is assumed.

For the critical stress we get

$$\sigma_{\text{krit}} = -\frac{F_{\text{krit}}}{\text{S}} = \frac{\pi^2 E J_y}{S l^2} . \qquad \text{(LM\_315)}$$

Alternatively, the following quantities are used.

- *radius of gyration*	$i = \sqrt{\dfrac{J_y}{S}}$,	(LM_316)
- *slendering ratio*	$\lambda = \dfrac{l}{i}$.	(LM_317)

Then, the critical stress could be expressed in the form $\sigma_{\text{krit}} = \dfrac{\pi^2 E}{\lambda^2}$ ... the relation is valid for $\lvert\sigma_{\text{krit}}\rvert \le \sigma_{\text{u}}$, where σ_{u} is the proportionality limit. (LM_318)

9.4.4. Four modes of buckling

In engineering, according to the types of constraints being applied, four modes of buckling are usually distinguished. They are depicted in Fig. LM_204.

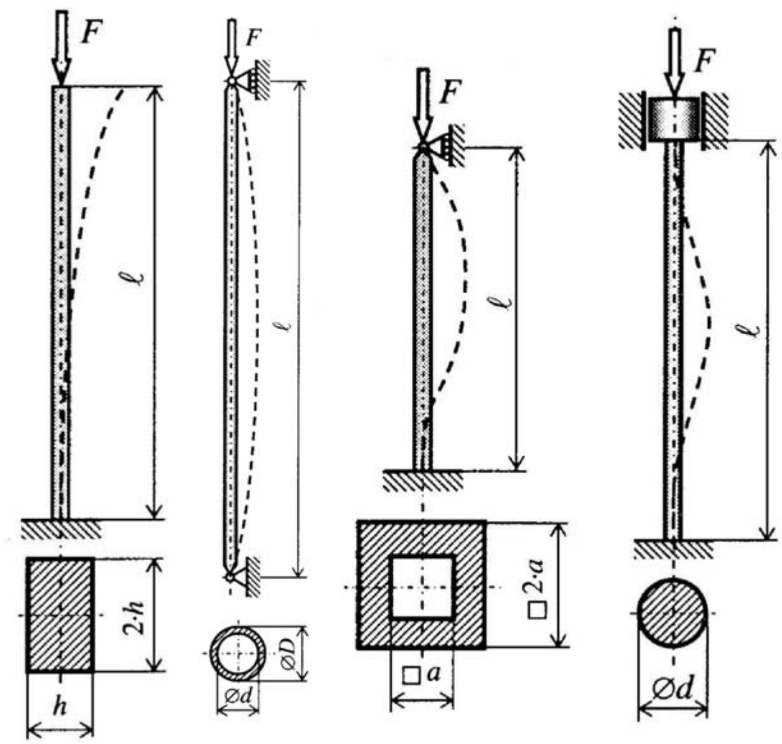


Fig. LM_204 ... Four buckling modes

The unified computational approach can be secured by defining four different values of n_i for $i = 1 \ldots 4$. They are presented in Table LM_20.

i	1	2	3	4
n_i	0.25	1	2	4

Table LM_20 ... Four parameters of buckling

Then, for the i^{th} bucking mode, the critical force can be expressed by

$$F_{\text{krit}}^{(i)} = n_i \pi^2 \frac{EJ_{\min}}{l^2}. \qquad \text{(LM\_319)}$$

Example – the first buckling mode, $i = 1, n_1 = 0.25$

Determine: Determine the dimension of a slender bar (strut) of a rectangular cross section $2h \times h$ made of steel with $E = 200$ MPa and the proportionality limit $R_u = 200$ MPa, and the yield limit $R_e = 250$ MPa. The strut is clamped, its opposite part is free and is loaded by the force $F = 2 \times 10^5$ N. The length is $l = 0.7$ m. See the leftmost subfigure of Fig. LM_204. Consider the safety factor $k = 3.5$. The critical force is

$$F_{\text{krit}}^{(1)} = kF = n_1 \pi^2 \frac{EJ_{\min}}{l^2}, \text{ where } n_1 = \frac{1}{4},\ J_{\min} = \frac{bh^3}{12} \cdots \frac{2h^4}{12}. \quad \text{(LM\_320)}$$

Then,

$$kF = n_1 \pi^2 \frac{E2h^4}{12l^2} \qquad \Rightarrow h = \sqrt[4]{\frac{kF12l^2}{2En_1\pi^2}} = \sqrt[4]{\frac{24kFl^2}{E\pi^2}}. \quad \text{(LM\_321)}$$

Substituting the input data we get $h = 44.6$ mm, approximately 45 mm.

Check the elastic behaviour. The critical stress for this mode is

$$\sigma_{\text{krit}} = \frac{F_{\text{krit}}^{(1)}}{\text{A}} = \frac{kF_{\text{krit}}^{(1)}}{2h^2} = \frac{3.5 \times 2 \times 10^5}{2 \times 45^2} = 172\,\text{MPa} < \text{R}_\text{u} = 200\text{MPa}. \quad \text{(LM\_322)}$$

So, the condition is satisfied and the suggested dimension, i.e. $h = 44.6$ mm, is acceptable.

Example – the second buckling mode, $i = 2, n_2 = 1$

Determine: The critical length l_{krit}, leading to a buckling instability of a steel tube loaded by an axial force $F = 1000$ N. Consider the safety factor $k = 3$. Assume that the strut is constrained by frictionless joints at both ends. The material constants are $E = 2.1 \times 10^5$ MPa, the proportionality limit $R_u = 190$ MPa and the yield limit $R_e = 230$ MPa. The outer and inner diameters of the tube are $D = 48.3$ mm and $d = 40.3$ mm respectively. See the second subfigure of Fig. LM_204.

The area is $A = \frac{\pi}{4}\left(D^2 - d^2\right) = 557\ \text{mm}^2$.

Quadratic moment is $J_{\min} = \frac{\pi}{64}\left(D^4 - d^4\right) = 8200\ \text{mm}^4$.

The critical force is

$$F_{\text{krit}}^{(2)} = \underbrace{kF = n_2 \pi^2 \frac{EJ_{\text{min}}}{l^2}}. \tag{LM\_323}$$

From the "braced" part of the previous equation we get,

$$kF = n_2 \pi^2 \frac{EJ_{\text{min}}}{l^2} \qquad \Rightarrow l = \sqrt{\frac{n_2 \pi^2 EJ_{\text{min}}}{kF}}. \tag{LM\_324}$$

Substituting the input data we get $l = l_{\text{krit}} = 2380\,\text{mm}$.

Check the elastic behaviour. The critical stress for this mode is

$$\sigma_{\text{krit}} = \frac{F_{\text{krit}}^{(2)}}{\text{A}} = \frac{kF_{\text{krit}}^{(2)}}{A} = \frac{3 \times 1000}{557} = 5.4\,\text{MPa} < \text{R}_{\text{u}} = 190\text{MPa}. \tag{LM\_325}$$

So, the computed length satisfies the prescribed stress conditions.

Example – the third buckling mode, $i = 3, n_3 = 2$

Determine: The allowed load value F_{D} for a steel strut with the Young modulus $E = 2.1 \times 10^5$ MPa, the proportionality limit $R_{\text{u}} = 200$ MPa and the yield limit $R_{\text{e}} = 240$ MPa. The strut is clamped; its other side is constrained by a shiftable joint support. Consider the safety factor $k = 3$. See the third subfigure of Fig. LM_204.

The area $A = 4a^2 - a^2 = 3a^2$.

The quadratic moment $J_{\text{min}} = J_2 - J_1$.

$J_1 = bh^3/12 \cdots b = 2a, h = 2a \cdots \dfrac{2a \times 8a^3}{12} = \dfrac{16a^4}{12}$.

$J_2 = bh^3/12 \cdots b = a, h = a \cdots \dfrac{a \times a^3}{12} = \dfrac{a^4}{12}$.

$J_{\text{min}} = \dfrac{15a^4}{12}$.

The critical force is

$$F_{\text{krit}}^{(3)} = n_3 \pi^2 \frac{EJ_{\text{min}}}{l^2}, \text{ where } n_3 = 2. \tag{LM\_326}$$

The allowed force is

$$F_{\mathrm{D}} = \frac{F_{\mathrm{krit}}^{(3)}}{\mathrm{k}} = n_3 \pi^2 \frac{EJ_{\mathrm{min}}}{kl^2}. \qquad \text{(LM\_327)}$$

Substituting the input data we get $F_{\mathrm{D}} = 38862\ \mathrm{N}$. To check the elastic behaviour, the following condition has to be satisfied

$$\sigma_{\mathrm{D}} = \frac{F_{\mathrm{D}}^{(3)}}{A} = \frac{kF_{\mathrm{krit}}^{(3)}}{A} < \mathrm{R_u} \cdots 190\mathrm{MPa}. \qquad \text{(LM\_328)}$$

Example – the fourth buckling mode, $i = 4,\ n_4 = 4$

Determine: The diameter d of a steel strut of the length $l = 1000\ \mathrm{mm}$. Material data are $E = 210\ \mathrm{MPa},\ R_{\mathrm{u}} = 210\ \mathrm{MPa},\ R_{\mathrm{e}} = 250\ \mathrm{MPa}$. The loading force is $5 \times 10^4\ \mathrm{N}$. Consider the safety factor $k = 4$. The constraints are depicted in the rightmost subfigure of Fig. LM_204.

The critical force is

$$F_{\mathrm{krit}}^{(4)} = \underbrace{kF = n_4 \pi^2 \frac{EJ_{\mathrm{min}}}{l^2}}_{(a)},\ \text{where}\ n_4 = 4. \qquad \text{(LM\_329)}$$

First, the quadratic moment J_{min} is determined from the part (a) of the previous equation

$$J_{\mathrm{min}} = \frac{kFl^2}{n_4 \pi^2 E}. \qquad \text{(LM\_330)}$$

then, realizing that $J_{\mathrm{min}} = \dfrac{\pi d^4}{64}$, we get the diameter

$$d = \sqrt[4]{\frac{64J_{\mathrm{min}}}{\pi}}. \qquad \text{(LM\_331)}$$

Substituting the input data we get $d = 38\ \mathrm{mm}$.

For more details see [7], [14], [17], [18], [19], [39].

10_EX. Simple examples

10.1 Tension – compression

Example – vertical rod loaded by its own weight and by an axial force

Given: The homogeneous prismatic rod is clamped at its upper end. The values of density ρ, the constant cross-sectional area S, the length l, and the gravitational acceleration g are known. At its lower end, the rod is loaded by a vertical force F. See Fig. EX_1.
Determine: The distribution of strain and stress along the rod. In this case, it is non-uniform.

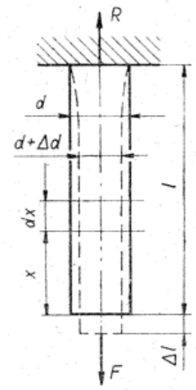


Fig. EX_1 ... Hanging rod

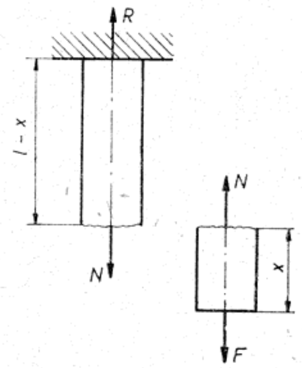


Fig. EX_2 ... Free body diagram

Free-body diagram allows determining the reaction force at the clamped area as

$$R = F + mg = F + \rho V g = F + \rho S l\, g\,, \tag{EX\_1}$$

where m is mass, V is volume, ρ is density, g is gravitational acceleration, l is length, and S is the cross-sectional area.

We are looking for deformations and forces occurring at a generic cross-sectional area displaced by the distance x measured from the lower end of the rod.

As explained before, in the text dedicated to the mechanics of rigid bodies, we will apply the *free body diagram* reasoning. See Fig. EX_2. The lower part of the rod is mentally removed and replaced by a force, say N, which is equivalent to forces acting in the upper part of the rod. Due to the continuous and homogeneous material distribution in the rod, the internal force N varies as a function of the x coordinate. So,

$N(x) = F + \rho g S x$, where Sx … volume, $\rho S x$ … mass, $\rho S x\, g$ … weight.

The dimensional check gives: $\dfrac{\text{kg}}{\text{m}^3}\dfrac{\text{m}}{\text{s}^2}\text{m}^2\text{m} = \dfrac{\text{kg}\,\text{m}}{\text{s}^2} = \text{N}$.

The stress as a function of x variable is

$$\sigma(x) = \frac{N(x)}{S} = \frac{F}{S} + \rho g x. \qquad \text{(EX\_2)}$$

The overall rod elongation Δl, and thus the strain ε, depends on material properties that are expressed by the so-called *constitutive relation*, i.e. by the relation between the stress and the strain. Its simple form is represented by a linear[1] function and is known as the *Hooke's law*. The coefficient of proportionality is denoted E and is called the Young's modulus. It is expressed by the same units as the stress, i.e. $\mathrm{N/m^2} = \mathrm{Pa}$. The usual value for the design steel is $E_{\mathrm{steel}} = 2.1 \times 10^{11}\ \mathrm{Pa}$.

In this case, the stress and the strain are functions of the x coordinate

$$\sigma(x) = E\varepsilon(x); \quad \varepsilon(x) = \frac{\Delta l(x)}{\mathrm{d}x}; \quad \sigma(x) = \frac{N(x)}{S}. \qquad \text{(EX\_3)}$$

Actually, the *local elongation*[2] is $\Delta l(x) = u(x)$ while the *total elongation* Δl is a cumulative quantity obtained by integration

$$\Delta l = \int_0^l \Delta l(x)\,\mathrm{d}x = \frac{1}{E}\int_0^l \sigma(x)\,\mathrm{d}x = \frac{1}{E}\int_0^l \left(\frac{F}{S} + \rho g x\right)\mathrm{d}x = \frac{1}{E}\left[\frac{F}{S}x + \rho g \frac{x^2}{2}\right]_0^l = \frac{1}{E}\left(\frac{Fl}{S} + \rho g \frac{l^2}{2}\right) = \Delta l_1 + \Delta l_2$$

... (EX_4)

Example – The above results could be applied to the analysis of the case of a long mining wire rope to which a cabin of a given weight is attached.

See the Matlab program **mpp_004e_elongation_of_mine_rope**

```
% mpp_004e_elongation_of_mine_rope
clear all; format compact
% sigma(x) = a + b*x
% a = F/S; b = ro*g
F= 10000;       % weight of cabin in N
L = 1000;       % length of rope in m
E = 2.1e11;     % Young modulus in Pa
ro = 7800;      % density in kg/m^3
g = 9.81;       % gravitational acceleration
d = 0.1;        % diameter of rope in m
S = pi*d^2/4;   % cross section in m^2
a = F/S; b = ro*g;
% elongation
deltaL1 = a*L/E             % elongation due to cabin's weight
deltaL2 = (b*L^2/2)/E       % elongation due to weight of rope
% deltaL = (a*L + b*L^2/2)/E
deltaL = deltaL1 + deltaL2  % total elongation
```

`deltaL1 = 0.0061` … elongation due to the cabin weight,
`deltaL2 = 0.1822` … elongation due to the rope weight,
`deltaL =  0.1882` … total elongation.

[1] The proportionality between the stress and the strain is not valid generally. It holds unless the plasticity behaviour of the material is reached. The details are treated later.

[2] That is the elongation of the element $\mathrm{d}x$.

The results are in [m]. The above example shows a rather exceptional engineering case when the own weight of the body is crucial and exceeds the effects of the external loading. In many engineering applications, the gravity effects might often be safely neglected.

If the rod, depicted in Fig. EX_1, is loaded by the force F only, (the weight of the rope is neglected) then the stress along the rod's length is constant and can be expressed in the form

$$\sigma = F / S \,. \tag{EX\_5}$$

Substituting the above relations for the stress and the strain we get

$$\sigma = E\varepsilon \quad \Rightarrow \quad \frac{F}{S} = E\frac{\Delta l}{L} \tag{EX\_6}$$

while the total elongation is

$$\Delta l = \frac{Fl}{ES} \,. \tag{EX\_7}$$

Finally, the strain, i.e. the relative deformation, is constant along the rod's length as well

$$\varepsilon = \frac{\Delta l}{l} = \frac{F}{ES} \,. \tag{EX\_8}$$

Example – rod with a variable cross-sectional area

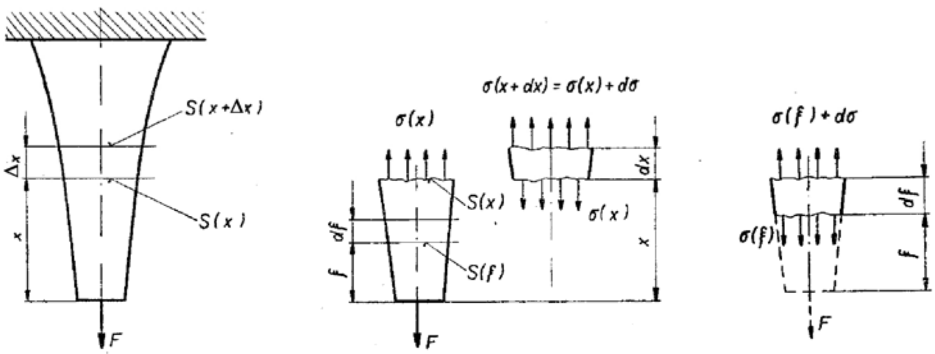


Fig. EX_3 ...Variable cross section

Consider a rod with a variable cross-sectional area, clamped at its upper end, depicted in Fig. EX_3 and Fig EX_5. The rod is loaded at its lower end by the force F. The length of the rod is l, the cross-sectional area is defined by a known continuous function $S = S(x)$. For two close cross sections, being Δx apart, we can express the equilibrium of forces in the form

$$S(x) + \Delta S = S(x) + \frac{\mathrm{d}S}{\mathrm{d}x}\Delta x \,. \tag{EX\_9}$$

The force increment ΔS is expressed using Taylor's series while the increments of higher orders are neglected. For stresses, using infinitesimal elements instead, we can write

$$\sigma(x) + \mathrm{d}\sigma = \sigma(x + \mathrm{d}x)\,. \qquad \text{(EX\_10)}$$

Using the free body diagram principles we can express the weight of the part of the rod being removed as

$$Q(x) = \rho g \int_V dV = \rho g \int_{\xi=0}^{x} S(\xi)\,\mathrm{d}\xi\ . \qquad \text{(EX\_11)}$$

The normal force and the corresponding stress in this cross section are

$$N(x) = F + Q(x)\,, \qquad \text{(EX\_12)}$$

$$\sigma(x) = \frac{N(x)}{S(x)} = \frac{F}{S(x)} + \frac{\rho g}{S(x)} \int_{\xi=0}^{x} S(\xi)\,\mathrm{d}\xi\ .$$

Knowing the cross-sectional area as a function of the length, the above relation could be evaluated.

The total elongation of the rod is

$$\Delta l = \int_{x=0}^{l} \varepsilon(x)\,\mathrm{d}x = \int_{x=0}^{l} \frac{\sigma(x)}{E}\,\mathrm{d}x\,. \qquad \text{(EX\_13)}$$

Neglecting the increments of higher orders, simplifying the notation by $\sigma = \sigma(x);\quad S = S(x)$, and realizing that the weight of the element is $\rho g S \mathrm{d}\xi$, then the equilibrium of forces acting on opposite sides of the element can be expressed in the form

$$
\begin{aligned}
&(\sigma + \mathrm{d}\sigma)(S + \mathrm{d}S) - \sigma S = \rho g S \mathrm{d}\xi \\
&\sigma S + \sigma \mathrm{d}S + \mathrm{d}\sigma S + \mathrm{d}\sigma \mathrm{d}S - \sigma S = \rho g S \mathrm{d}\xi \\
&\sigma \mathrm{d}S + \mathrm{d}\sigma S = \rho g S \mathrm{d}\xi \quad \cdots \quad \text{increments of higher orders were neglected} \qquad \text{... (EX\_14)} \\
&\mathrm{d}(\sigma S) = \rho g S \mathrm{d}\xi \\
&\frac{\mathrm{d}(\sigma S)}{\mathrm{d}\xi} = \rho g S
\end{aligned}
$$

Example – rod (rope) of the same strength, part 1

The idea is to define a rod with such a variable cross-sectional area that would have the same stress along its length. Suppose that the allowable stress σ_{AL} is known. How to find a function $S(x)$ satisfying the above criteria? We might start with equilibrium considerations

$$\sigma \mathrm{d}S + \mathrm{d}\sigma S = \rho g S \mathrm{d}\xi \qquad \text{(EX\_15)}$$

and realize that if the stress σ should be constant, then its increment has to be zero, so $\mathrm{d}\sigma = 0$. From now on, the required constant stress is just the allowable stress, so

$$\begin{aligned} &\sigma_{\mathrm{AL}}\,\mathrm{d}S = \rho g S \mathrm{d}\xi, \\ &\frac{dS}{S} = \frac{\rho g}{\sigma_{\mathrm{AL}}}\,\mathrm{d}\xi. \end{aligned} \qquad \text{(EX\_16)}$$

Integrating along the length of the rod (rope), and considering that the lower cross-sectional area is S_0, we get

$$\int_{S_0}^{S} \frac{\mathrm{d}S}{S} = \frac{\rho g}{\sigma_{\mathrm{AL}}} \int_{\xi=0}^{x} \mathrm{d}\xi\,, \qquad \text{(EX\_17)}$$

$$\begin{aligned} &\lg \frac{S}{S_0} = \frac{\rho g}{\sigma_{\mathrm{AL}}}\,x, \\ &S = S(x) = S_0 \exp\left(\frac{\rho g}{\sigma_{\mathrm{AL}}}\,x\right). \end{aligned} \qquad \text{(EX\_18)}$$

Example – rope of the same strength, part 2

Given: A rope of the length $L = 3000\,\mathrm{m}$, diameter at the lower end $d_0 = 0.1\,\mathrm{m}$, density $\rho = 7800\,\mathrm{kgm}^{-3}$, Young's modulus $E = 2.1\times 10^{11}\,\mathrm{Pa}$, gravitational acceleration $g = 9.81\,ms^{-2}$, weight of the cabin attached at the lower end of the rope $F = 10000\,\mathrm{N}$, allowable stress $\sigma_{\mathrm{AL}} = 1\times 10^{8}\,\mathrm{Pa}$.
Determine: Elongation of the rope due to its own weight, elongation due to the weight of the cabin, diameter of the rope as a function of its length.

The elongation due to own weight

$$\Delta l_1 = \frac{1}{E}\int_0^l \sigma(x)\,\mathrm{d}x = \frac{1}{E}\int_0^l \sigma_{\mathrm{d}}\,\mathrm{d}x = \frac{\sigma_{\mathrm{d}} l}{E} \quad \text{... in our case, we get } \texttt{1.4286}\text{ m.} \qquad \text{(EX\_19)}$$

The elongation and the cross-sectional area as a the function of the length due to the weight of the cabin

$$\Delta l_2 = \frac{1}{E}\int_0^l \frac{F}{S(x)}\,\mathrm{d}x; \quad S(x) = S_0 \mathrm{e}^{ax}; \quad a = \frac{\rho g}{\sigma_{\mathrm{d}}}; \quad S_0 = \frac{F}{\sigma_{\mathrm{d}}}, \qquad \text{(EX\_20)}$$

$$\Delta l_2 = \frac{F}{ES_0}\int_0^l \frac{1}{\mathrm{e}^{ax}}\,\mathrm{d}x = \frac{F}{ES_0}\int_0^l \mathrm{e}^{-ax}\,\mathrm{d}x = \frac{F}{ES_0}\frac{-1}{a}[\mathrm{e}^{ax}]_0^l = \frac{F}{ES_0}\frac{1}{a}[\mathrm{e}^{ax}]_l^0 = \frac{F}{ES_0}\frac{1}{a}\left(1 - \mathrm{e}^{-al}\right). \qquad \text{(EX\_21)}$$

In our case we get `0.0071` m which is a negligible value with respect to the elongation due to the weight of the cabin

How to compute it shows the Matlab program **mpp_002e_rope_of_equal_strength.** The results are presented in Fig. EX_4.

```
% mpp_002e_rope_of_equal_strength
clear all
sigd = 1e8;       % allowed strength in Pa, i.e. N/m^2
d0 = 0.1;         % initial diameter in m
S0 = pi*d0^2/4;   % initial cross section in m^2
g = 9.81;         % gravity acceleration in m/s^2
ro = 7800;        % density in kg/m^3

x = 0:3000;       % the length
Sx = S0*exp(ro*g*x/sigd);       % cross section as a function of x
dx = sqrt(4*Sx/pi);             % diameter = f(x)
rx = dx/2;                      % radius

% elongation of the rope of the length 3000 m - due to gravity only
L = 3000;
E = 2.1e11;
a = ro*g/sigd;
F = 10000;  % the cabin weight in N
sig0 = F/S0;

deltaL1 = sigd*L/E                  % influence of gravity
deltaL2 = F*(1- exp(-a*L))/(E*S0*a) % influence of the cabin's weight

y1 = sigd*x/E;
y2 = F*(1 - exp(-a*x))/(E*S0*a);

figure(1)
subplot(1,2,1)
plot(x,rx, 'linewidth', 2)
title('rope of equal strenght', 'fontsize', 16)
ylabel('radius in [m]', 'fontsize', 16);
xlabel('length in [m]', 'fontsize', 16);
subplot(2,2,2)
plot(x,y1, 'linewidth',2)
title('elongation [m]', 'fontsize', 16)
ylabel('due to gravity', 'fontsize', 16);
subplot(2,2,4)
plot(x,y2, 'linewidth',2)
ylabel('due to cabin''s weight', 'fontsize', 16);
xlabel('length in [m]', 'fontsize', 16)
```

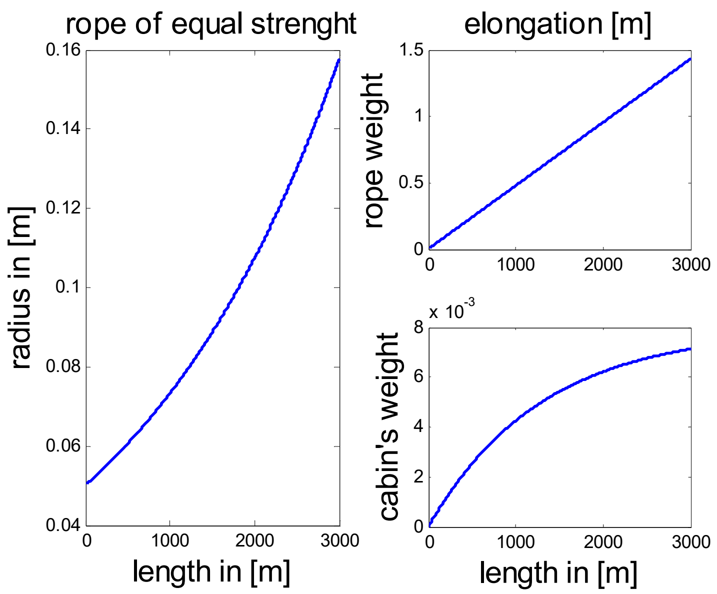


Fig. EX_4 ... Radius of the rope cross-sectional area as a function of length, elongations

Example – rotating arm

Given: The arm of prescribed dimensions, see Fig. EX_6, rotates by the constant angular speed ω, density ρ, cross-sectional area S.
Determine: Using free body diagram and d'Alembert principle determine displacement, force and stress within the body.

Apparent inertia force acting on the indicated mass element $\mathrm{d}m = \rho\,\mathrm{d}V = \rho S \mathrm{d}\xi$ is $\mathrm{d}m\xi\omega^2$, i.e. mass$\times$radius$\times$angular velocity squared. It is in equilibrium with the internal force $N(x)$.

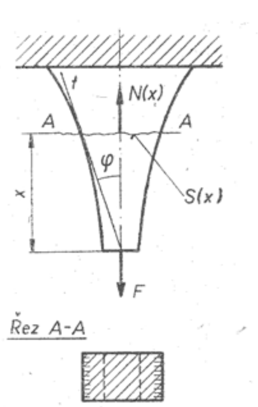


Fig. EX_5 ... Varying cross section

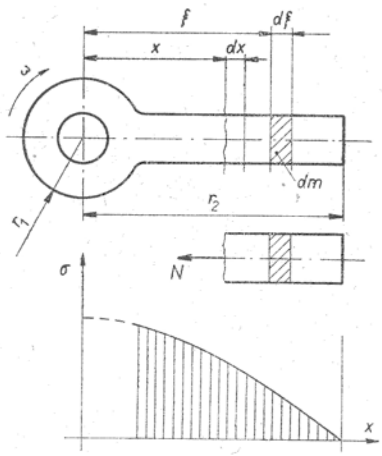


Fig. EX_6 ... Rotating arm

The internal force acting at the cross section determined by the coordinate x is

$$N(x) = \rho S\omega^2 \int_{\xi=x}^{r_2} \xi \mathrm{d}\xi = \frac{1}{2}\rho S\omega^2 \,(r_2^2 - x^2)\,. \qquad \text{(EX\_22)}$$

So, the stress as a function of the x-coordinate is

$$\sigma(x) = \frac{N(x)}{S} = \frac{1}{2}\rho\omega^2 (r_2^2 - x^2)\,. \qquad \text{(EX\_23)}$$

According to Hooke's law the local elongation of the element of the elementary length $\mathrm{d}x$ is

$$\Delta \mathrm{d}x = \frac{\sigma(x)}{E}\mathrm{d}x = \frac{\rho\omega^2}{2E}(r_2^2 - x^2)\mathrm{d}x\,. \qquad \text{(EX\_24)}$$

The total elongation of the arm is obtained by integration

$$\Delta l = \int_0^{r_2} \Delta\,\mathrm{d}x = \frac{\rho\omega^2}{2E}\int_{x=0}^{r_2}(r_2^2 - x^2)\mathrm{d}x = \frac{\rho\omega^2}{2E}[r_2^2 x - \frac{x^3}{3}]_0^{r_2} = \frac{\rho\omega^2}{2E}\frac{2}{3}r_2^3 = \frac{\rho\omega^2}{3E}r_2^3\,. \qquad \text{(EX\_25)}$$

The obtained expression is approximate since the presented analysis does not take into account the non-uniform state of deformation in the vicinity of the central hub.

Example – a thin prismatic ring of the cross-sectional rectangular area $S = bh$ is loaded by the internal pressure p

Given: Radius r, thickness h, width b, pressure p. See Fig. EX_7.
Determine: Stress, strain, radial elongation.

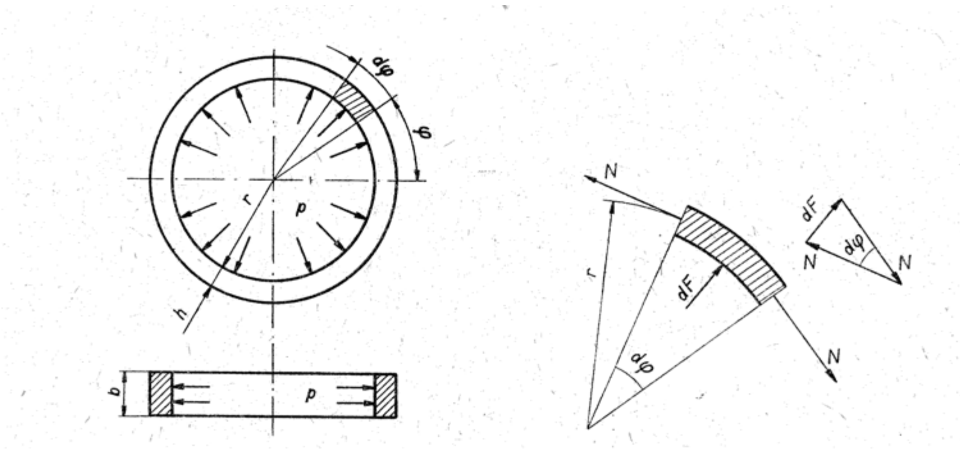


Fig. EX_7 ... A thin ring

Use the free body diagram principle and observe the equilibrium of forces acting on the element $hb\,\mathrm{d}\varphi$ depicted in Fig. EX_7.

The equilibrium conditions for the element, determined by a small elementary angle $\mathrm{d}\varphi$, give the radial force in the form

$$\mathrm{d}F = N\,\mathrm{d}\varphi, \qquad \text{(EX\_26)}$$

since the small circle can be approximated by a straight line. This elementary force could also be expressed by means of a product of the pressure and the elementary surface as

$$\mathrm{d}F = pbr\,\mathrm{d}\varphi. \qquad \text{(EX\_27)}$$

Combining the last two equations we get

$$N = pbr. \qquad \text{(EX\_28)}$$

Assuming a uniform distribution of the force N within the cross-sectional area we might express the stress as

$$\sigma = \frac{N}{S} = \frac{N}{bh} = \frac{pbr}{bh} = p\frac{r}{h}. \qquad \text{(EX\_29)}$$

This is the circumferential or tangential stress. We found that the radial pressure inside the ring evokes the tensional circumferential stress in the ring cross-section. Under these simplifying conditions (a thin ring, i.e. $r >> h$) the radial stress in the ring – that evidently has to be there as well – is not accounted for. The analysis of thick-walled vessels will give a more detailed answer.

Assuming a uniform circumferential strain distribution within the cross-sectional area we get

$$\varepsilon = \frac{2\pi(r+\Delta r) - 2\pi r}{2\pi r} = \frac{\Delta r}{r} = \frac{\sigma}{E} = \frac{pr}{Eh}. \qquad \text{(EX\_30)}$$

Due to the internal pressure p the radius of the ring is increased by

$$\Delta r = \frac{pr^2}{Eh}. \qquad \text{(EX\_31)}$$

Example – a rotating thin ring

Given: A thin ring rotates by a constant angular velocity ω. In Fig. EX_7 disregard the internal pressure. In this case the measure b is more important.
Determine: The circumferential stress due to the ring rotation.

The apparent inertia force – the centrifugal force in this case – acting on the mass element is

$$\mathrm{d}F = \mathrm{d}m r\omega^2 = \rho\,\mathrm{d}V r\omega^2 = \rho h b r\,\mathrm{d}\varphi\, r\omega^2. \qquad \text{(EX\_32)}$$

As in the previous case, the tensional circumferential force is

$$N = \frac{\mathrm{d}F}{\mathrm{d}\varphi} = \rho h b\, r^2\omega^2. \qquad \text{(EX\_33)}$$

Realizing that the circumferential velocity is $v = r\omega$, the circumferential stress is

$$\sigma = \frac{N}{bh} = \rho v^2. \qquad \text{(EX\_34)}$$

This approach could be used for an approximate determination of the circumferential stress in the rotating flywheel rim.

10.2. The strain energy and the work exerted by an external force – uniaxial case

It is taken for granted that in statics the time quantity plays no role – there are no accelerations, no inertia forces. The actual loading process, however, always occurs in time but in statics, we overcome this inconsistency by stating that the loading process is so slow that the inertia forces could be neglected.

Here, for the proper understanding of the loading process, we will temporarily deal with the time dimension. Imagine a slender rod of the length l and of the cross-sectional area S being loaded by an axial force $F(t)$ that is a function of time. The initial value of that force at time $t = 0$ is zero, then it raises to its maximum value, say $F_{\max}$. As said before, the loading process is assumed to be so slow that the inertia forces could be neglected.

Let's introduce a new quantity λ, varying in the range $\langle 0,1\rangle$, as the time-dependent ratio of the immediate to the maximum value of force in such a way that $\lambda(t) = \dfrac{F(t)}{F_{\max}}$.

From it follows that

$$F(t) = F_{\max}\lambda(t). \qquad \text{(EX\_35)}$$

Under these assumptions, the elongation of the rod as a function of time could be expressed by

$$u(t) = \frac{\lambda(t)F_{\max}l}{ES} = \lambda(t)\Delta l, \qquad \text{(EX\_36)}$$

where Δl is the total elongation corresponding to the maximum force $F_{\max}$.

Also, the *elementary work of external force* is a function of time. Thus,

$$\mathrm{d}W(t) = F(t)\,\mathrm{d}u = \lambda(t)F_{\max}\Delta l\,\mathrm{d}\lambda(t) = F_{\max}\Delta l\lambda\mathrm{d}\lambda. \qquad \text{(EX\_37)}$$

The *total (cumulative) work of external force* – from the beginning of loading to its end – is obtained by the following integration

$$W = F_{\max}\Delta l\int_0^1 \lambda\mathrm{d}\lambda = \frac{1}{2}F_{\max}\Delta l. \qquad \text{(EX\_38)}$$

Assuming the validity of Hooke's law – ($\Delta l = F_{\max}l/ES$) – the previous relation could be rewritten into the form

$$W = \frac{F_{\max}^2 l}{2ES}. \qquad \text{(EX\_39)}$$

10.3. Statically indeterminate cases

In courses dedicated to mechanics of rigid (non-deformable) bodies, we have stated that a structure is *statically indeterminate* when the static equilibrium equations are insufficient for determining the internal forces and reactions acting on that structure. The mechanics of deformable bodies (strength of material) is able to solve these tasks by adding a sufficient number of so-called *deformable conditions*.

Example – statically indeterminate clamped rod

Given: Length $l = a + b$, force F, cross-sectional area S, Young modulus E. There are no axial gaps between the rod and its supports. See Fig. EX_8.
Determine: Reactions R_1, R_2.

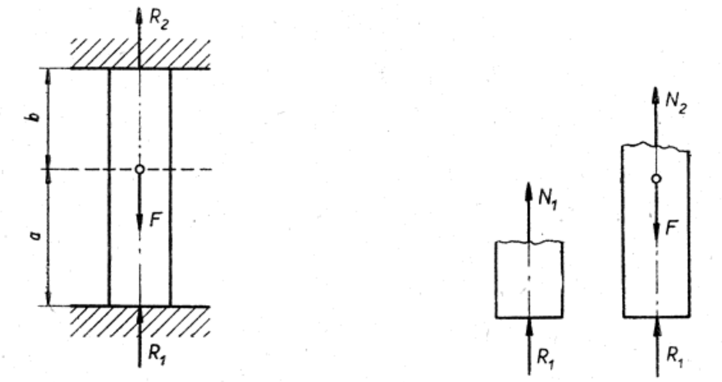


Fig. EX_8 ... Statically indeterminate clamped rod

The clamping reactions due to the loading by the force F are R_1, R_2 respectively. From the point of view of the mechanics of rigid bodies only one equilibrium equation is available, i.e.

$$R_1 + R_2 - F = 0. \tag{EX\_40}$$

The equation contains two unknowns – so the tools of the mechanics of rigid bodies do not suffice to solve the task. We say that the task is *statically indeterminate*. Living in the world of the mechanics of deformable bodies we can add a so-called *deformable condition* (also called the condition of compatibility of deformations). In this, case it represents the condition that the length of the rod l cannot change due to the loading since the supports are assumed to be perfectly stiff. A free body diagram is sketched for two cases – in a cross section below the acting point of the force F and above of it. The internal forces N_1, N_2 represent actions of the "removed" parts.

For the shorter part of the rod – below the force F – the equilibrium condition is $N_1 + R_1 = 0$, while for the longer one, it is $R_1 + N_2 - F = 0$.

So, the internal forces are

$$N_1 = -R_1, \quad N_2 = F - R_1. \tag{EX\_41}$$

The overall elongation of the rod (which has to be zero in this case) is composed of individual non-zero deformations of two parts of the rod with lengths *a* and *b* respectively. So,

$$\Delta l = \Delta a + \Delta b = \frac{N_1 a}{ES} + \frac{N_2 b}{ES} = \frac{-R_1 a}{ES} + \frac{(F - R_1)b}{ES} = -\frac{R_1(a+b)}{ES} + \frac{Fb}{ES}. \tag{EX\_42}$$

The clamping supports are assumed to be perfectly stiff, so the rod's elongation $\Delta l = 0$. From it follows that the above deformation condition could be expressed in the form

$$-R_1(a+b)+Fb=0. \qquad \text{(EX\_43)}$$

Now, there are two available equations

$$\begin{aligned} R_1 + R_2 &= F, \\ R_1(a+b) &= Fb. \end{aligned} \qquad \text{(EX\_44)}$$

Solving them we get

$$R_1 = \frac{b}{a+b}F; \quad R_2 = F - R_1 = F - \frac{b}{a+b}F = F(1-\frac{b}{a+b}) = F\frac{a+b-b}{a+b} = F\frac{a}{a+b}. \qquad \text{(EX\_45)}$$

Furthermore, we can determine, how the point of action of the force F is displaced

$$\Delta b = \frac{N_2 b}{ES} = \frac{R_2 b}{ES} = F\frac{a}{a+b}\frac{b}{ES} = \frac{F}{ES}\frac{ab}{a+b}. \qquad \text{(EX\_46)}$$

Example – statically indeterminate truss structure

Given: Structure depicted in Fig. EX_9, length a, force F.
Determine: The forces in P_1, P_2, P_3 and the displacements of the joint A, i.e. u, v.

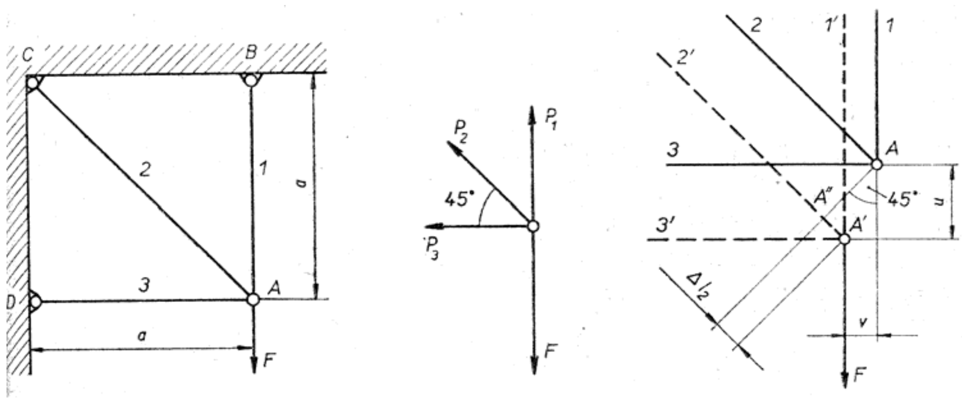


Fig. EX_9 ... Indeterminate truss structure

Assume that the rods are of the same materials and have the same cross-sectional areas. The rods are connected by frictionless joints, and there is no initial pre-stress. The structure is loaded by a single vertical force F acting at the joint A.

Generally, only two equilibrium equations could be written for a system of forces passing through a single point in the plane. In this case we have

$$\begin{aligned} P_1 + P_2 \sin 45° - F &= 0, \\ P_2 \cos 45° + P_3 &= 0. \end{aligned} \qquad \text{(EX\_47)}$$

In these two equations, there are three unknowns. The missing equation can be obtained from the deformation condition – in this case, it represents the fact that the resulting displacements of individual rods, ending at the joint A, have to be identical.

As always, small deformations are considered – from it follows that the small rod's rotations due to the applied loading are neglected. Under these approximations the deformed configuration of the structure is plotted in Fig. EX_9 by dashed lines. The joint A moves to a new position A' defined by displacements u and v respectively.

The elongations of individual rods are

$$\begin{aligned} \Delta l_1 &= u, \\ \Delta l_2 &= u \sin 45° - v \cos 45°, \\ \Delta l_3 &= -v \quad \text{... the rod is actually shortened.} \end{aligned} \qquad \text{(EX\_48)}$$

Eliminating u and v from the above equations and rearranging we get the deformation condition in the form

$$\begin{aligned} \Delta l_2 &= \Delta l_1 \frac{\sqrt{2}}{2} + \Delta l_3 \frac{\sqrt{2}}{2}, \\ \sqrt{2}\Delta l_2 &= \Delta l_1 + \Delta l_3. \end{aligned} \qquad \text{(EX\_49)}$$

Assuming the validity of Hooke's law the elongations of rods are

$$\Delta l_1 = \frac{P_1 a}{ES} \quad \Delta l_2 = \frac{P_2 a \sqrt{2}}{ES} \quad \Delta l_3 = \frac{P_3 a}{ES}. \qquad \text{(EX\_50)}$$

Using the previously derived deformation condition, we get the missing equation in the form

$$2P_2 = P_1 + P_3. \qquad \text{(EX\_51)}$$

Summarizing, the required equations are

$$\begin{aligned} &P_1 + P_2 \frac{\sqrt{2}}{2} - F = 0, \\ &P_2 \frac{\sqrt{2}}{2} + P_3 = 0, \\ &2P_2 = P_1 + P_3. \end{aligned} \qquad \text{... (EX\_52)}$$

In the matrix form we have

$$\begin{bmatrix} 1 & \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \\ 0 \end{Bmatrix}. \qquad \text{(EX\_53)}$$

How to proceed in Matlab? See the program `mpp_006e_three_rods`.

```
% mpp_006e_three_rods
clear all
sq = sqrt(2)/2; K = [1 sq 0; 0 sq 1; 1 -2 1];

invK = inv(K); rhs = [1 0 0]';

P = invK*rhs
```

Notice, that the force F in the program was considered to be equal to 1. Then,

```
 0.7929     0.2929      -0.2071
```

To find the response for the actual loading, it suffices to pre-multiply the previous 'normalized' results by the actual force.

Matlab could help to solve the problem symbolically to get the result in an analytic form as shown in the program `mpp_007e_three_rods_sym`.

```
% mpp_007e_three_rods_symb
clear all
syms K s P F

s = sym(sqrt(2)/2)
F = [F 0 0].'

K = [1 s 0; 0 s 1; 1 -2 1]
invK = inv(K);
P = invK*F
```

The result is

```
P =

 1/2*(2^(1/2)+4)/(2+2^(1/2))*F
                 1/(2+2^(1/2))*F
    -1/2*2^(1/2)/(2+2^(1/2))*F
```

This could be expressed in a 'nice human' form as

$$P_1 = \frac{1+2\sqrt{2}}{2(1+\sqrt{2})} F, \quad P_2 = \frac{\sqrt{2}}{2(1+\sqrt{2})} F, \quad P_3 = \frac{1}{2(1+\sqrt{2})} F. \qquad \text{(EX\_54)}$$

10.4. Thermal stress

A tendency of a body to change its geometrical shape due to the change of temperature is called the *thermal expansion*. If a body is mechanically constrained in such a way that it cannot freely dilate when the temperature is changing, then an additional stress occurs in the body even if no external loading is applied. This happens for statically indeterminate cases – the suppressed dilatation evokes the thermal strain and consequently the thermal stress.

Example – heated statically indeterminate rod

Given: A thin rod, of the length $l = a + b$ with cross-sectional areas S_1, S_2 and Young's modulus E, is put – at a given temperature and with no axial clearance – in between two supports as seen in Fig. EX_10. Then, the temperature is increased by Δt degrees.
Determine: The reaction R.

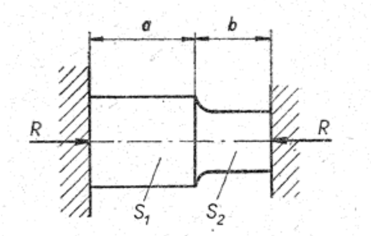


Fig. EX_10 ... Statically indeterminate rod.

If the rod were free (unconstrained) – then due to the temperature increase by Δt – it would increase its length by $\Delta l = \alpha \, l \Delta t$, where $\alpha \, [1/\deg]$ is the *coefficient of thermal expansion*. Due to the existence of perfectly stiff supports, the rod's elongation is suppressed and consequently the reaction forces, say R, are induced – they are of the same magnitude but of opposite directions.

The elongations of individual parts of the rod consist of two parts – the elongation due to the temperature increase and the contraction due to the loading by compressive reaction forces. Thus,

$$\Delta a = \alpha a \Delta t - \frac{Ra}{ES_1} \qquad \Delta b = \alpha b \Delta t - \frac{Rb}{ES_2}. \tag{EX\_55}$$

Substituting these expressions into the deformation condition, requiring that the overall deformation is zero, i.e. $\Delta l = \Delta a + \Delta b = 0$, we get the unknown reaction in the form

$$R = \frac{(a+b) S_1 S_2 E \alpha \Delta t}{a S_2 + b S_1}. \tag{EX\_56}$$

Assuming the validity of Hooke's law, the corresponding stresses in the rod's parts are

$$\sigma_1 = -\frac{R}{S_1} = \frac{(a+b) S_2 E \alpha \Delta t}{a S_2 + b S_1}, \quad \sigma_2 = -\frac{R}{S_2} = \frac{(a+b) S_1 E \alpha \Delta t}{a S_2 + b S_1}. \tag{EX\_57}$$

For current design steels the coefficient of the thermal expansion is $\alpha = 1.2 \times 10^5 \deg^{-1}$.

Lot of examples could be found in [21], [39].

11_FE. A brief survey of finite element method

11.0. Introduction

Finite element method is a well-established procedure, routinely used in mechanical engineering by means of commercial finite element packages. This text should help to understand the basics and feel the flavor of the method and might help to realize what is behind seductive color screens of those packages, full of rolled-out menus containing – for a beginner – a lot of often unknown choices. There are a lot of publications recommended for future studies. For example [3] to [8], [11] to [14], [18], [20], [25] to [28], [30], [33], [34], [42].

11.1. Discretization of continuous quantities in continuum mechanics

Modelling a dynamic rigid body system leads to equations of motion having the form of the system of ordinary differential equations. Such systems have the finite number of eigenfrequencies and eigenmodes.

Structural elements appearing in engineering practice (rods, beams, plate, shells, etc.) are, however, not rigid, but generally flexible, having continuously distributed stiffness and mass. Their mathematical descriptions lead to partial differential equations. Corresponding frequency equations are of a transcendent type and give the infinite number of degrees of freedom and thus the infinite number of frequencies. See [23].

Rigid model systems are thus a simpler representation of reality, than continuous models, they are, on the other hand, easier to solve. Continuous models are thus a better representation of reality but at a cost. They are more difficult to solve, which leads us to the digitization again – but of a different kind, that will be explained in the following paragraphs.

11.2. Basic equations of continuum mechanics

Continuum mechanics deals with deformations of bodies and forces that are responsible for those deformations. We will limit our attention to solids, assuming that the material properties are independent on dimensions of an investigated specimen. Also, the material quantities and those describing the deformation process are assumed to be continuous functions of space and time.

The motion and deformations of a solid continuous body are described by three systems of equations.

11.3. Cauchy equations of motion

relating inner, outer and inertia forces, have the form

$$\frac{\partial \, {}^t_t\sigma_{ij}}{\partial \, {}^t x_j} + {}^t f_i = {}^t\rho \, {}^t\ddot{x}_i \, . \qquad \text{(FE\_1)}$$

These equations have, however, a different interpretation in linear and non-linear concepts of the world. Generally, in continuum solid mechanics with finite displacement, rotations and strains, the stress tensor ${}^t_t\sigma_{ij}$ (Cauchy or true stress) is measured in the current configuration at time t (this is indicated by the upper left-hand side index) and is related to the same configuration at the same time *t* (indicated by the lower left-hand side index). The Cauchy (true) stress tensor is thus defined as the current elementary force acting on the current elementary surface in its deformed shape. Elementary forces ${}^t f_i$ and the density ${}^t\rho$ are also related to the current configuration.

This seems to be obvious, but in linear mechanics (where infinitesimal displacements and strains are assumed) we are employing a simplified approach and instead of the Cauchy (true) stress we work with so-called engineering stress – ${}^t_0\sigma_{ij} \approx \sigma_{ij}^{\mathrm{eng}}$ (usually we write only σ_{ij}), which relates the current elementary force to the not to the current but to the initial (reference) configuration considered at time $t = 0$. In the same, i.e. initial configuration are considered the inner forces and the density, i.e. ${}^0 f_i = f_i$, ${}^0\rho = \rho$. Also, the coordinates are considered in reference configuration only, i.e. ${}^t x_i \approx {}^0 x_i$ – which is briefly denoted x_i.

11.4. Kinematic relations

relate strains and displacements and secure thus compatibility conditions. There is an infinite number of ways how to define strain tensors. As an example, let's present the Green-Lagrange strain tensor, which is independent of the choice of coordinate system as well as of rigid body rotations.

$$\varepsilon_{ij}^{\mathrm{GL}} = \frac{1}{2}\left(\frac{\partial\, {}^t u_i}{\partial\, {}^0 x_j} + \frac{\partial\, {}^t u_j}{\partial\, {}^0 x_i} + \frac{\partial\, {}^t u_k}{\partial\, {}^0 x_i}\frac{\partial\, {}^t u_k}{\partial\, {}^0 x_j}\right). \tag{FE\_2}$$

This tensor – in case of small deformations and strains – simplifies into a so-called engineering (or infinitesimal) strain tensor having the form

$$\varepsilon_{ij}^{\mathrm{eng}} = \frac{1}{2}\left(\frac{\partial\, {}^t u_i}{\partial\, {}^0 x_j} + \frac{\partial\, {}^t u_j}{\partial\, {}^0 x_i}\right). \tag{FE\_3}$$

11.5. Constitutive equations

relate stresses and strains. Generally, any couple of energetically conjugate stress and strain quantities could be employed. In a linear case, the generalized Hooke's law, with engineering stress and Cauchy's infinitesimal strain, is being employed, i.e. $\sigma_{ij}^{\mathrm{eng}} = C_{ijkl}\,\varepsilon_{kl}$.

Impossibility to solve the system of partial differential equations for complicated geometrical cases and for generic initial and boundary conditions led to the development of numerical methods.

11.6. Numerical approaches

It was probably the *finite difference method* which was primarily used for the solution of partial differential equations describing the solid continuum problems. Partial derivatives appearing in these equations were systematically replaced by finite differences.

For example, the second derivatives can be replaced by central differences according to the following formula

$$\frac{\partial^2 y(x)}{\partial x^2} = \frac{1}{h^2}\left[y(x-h) + 2y(x) + y(x+h)\right] + O(h^2) \qquad \text{(FE\_4)}$$

where h is a parameter denoting the mesh size and the term $O(h^2)$ represents the residuum, showing the order of error. For more details see [11], [12b], [41].

Employing linear approximations and other simplifying assumptions allowed to find closed-form solutions describing the mechanical behavior of simple engineering design parts, like bars, beams, plates, etc. This way, the relations between generalized coordinates, displacements, and forces in nodes connecting these parts were expressed. The approach, based on these ideas, is known as the theory of transfer matrices. A nice introduction can be found in [4], [30].

What followed was the so-called *direct stiffness method* that was based on the idea of decomposing the structure into the assembly of simple design parts (again bars, beams, plates, etc.). Then, for each part, called element (e), were defined generalized forces $Q^{(e)}$ and generalized displacements $q^{(e)}$ related by means of so-called elementary stiffness matrices $\mathbf{k}^{(e)}$.

Then, the elementary forces, displacements and stiffness matrices were assembled in such a way as to create the global quantities, $\mathbf{Q}, \mathbf{q}, \mathbf{K}$, describing the overall behavior of the considered structure. Finally, the system of algebraic equations $\mathbf{Kq} = \mathbf{Q}$, was solved determining the unknown displacements. One of the forefathers of this method was professor John Argyris. It was during the Second World War.

Approximately at the same time, a similar method was conceived in the U.S.A. It was given the name *matrix displacement method*. Besides, the elements derived by the direct stiffness method the new elements, based on the continuum mechanics considerations were conceived, namely triangular, rectangular, brick and other elements. The equations of motion were derived using variational principles. In the mentioned form the above method was already very close to what is known under the name the deformation variant of finite element method, where the displacements are considered to be primary unknowns form which the force variables are consequently evaluated. See [30].

There exists a complementary formulation, namely the *force variant of finite element method*, where the forces are considered to be primary unknowns and the displacements are computed from them. The force formulation, based on the Castigliano theorem, did not gain such popularity as the deformation one. This was due to the fact that the resulting algorithm of force variant depends on the correct determination of the *static indeterminacy* of the solved system. See [30].

The *Hybrid formulations of the finite element method* are based on the combinations of deformation and force variants. The stresses are approximated within the elements, while the displacement on their surfaces. See [30].

The *Boundary integral method* is just another tool for the numerical solution of continuum mechanics problems, being based on variational principles. The method, while satisfying the internal equilibrium conditions, allows solving the problem on the surface of the body only. See [5], [8].

In the following text, we will concentrate our attention on the *deformation variant of the finite element method* within the scope of solid continuum mechanics. In order to better understand the method's nonlinear features presented in the following text, we will start with its linear background here.

The *Finite Element Method* (FEM) is based on the discretization of the solid mechanics tasks in space and time. In space we fill-up the volume occupied by the considered body (or bodies) by many small parts (called elements) of simple geometrical shapes whose inertia, damping and stiffness properties are known and expressed in matrix forms.

In time, we give up to find the response of the discretized body as a continuous function of time. Instead, we aim to express all the geometrical and force variables at discrete time periods, whose time distance – called time step – is rather small.

Satisfying compatibility conditions and equations of motion with prescribed boundary and initial conditions, we are then able to determine the response of the body in kinematic and force quantities at all parts of the body and at all considered time instants. See [4], [11].

There are many ways how to derive and explain the basic relations for the FEM. The one presented here is based on the principle of virtual work.

11.7. The finite element method

The principle of the virtual work, formulated for solid continuum mechanics, states, see [23], that the virtual work done by internal forces ∂U is equal to that done by external forces ∂W, so

$$\partial U = \partial W \,. \qquad \text{(FE\_5)}$$

The idea behind that thought experiment is, that all the particles (material points) of the body are subjected to the virtual displacements $\partial \mathbf{u}$, while the time, for a given moment, is frozen. It is also assumed that the acting forces and boundary conditions do not change during that virtual displacement. To prescribed virtual displacements are uniquely assigned the corresponding virtual strains $\partial \boldsymbol{\varepsilon}$.

As mentioned before, to clarify the presentation, the explanation will proceed in two steps – first for linear and then for the non-linear case. This allows pinpointing the differences and similarities.

11.8. Linear case

Let's start with small strains and small deformation using the usual engineering notation. The energy balance according to Eq. (FE_5) – for a considered body in Fig. FE_1 – is

$$\int_V \partial\boldsymbol{\varepsilon}^{\mathrm{T}} \boldsymbol{\sigma}\, \mathrm{d}V = \int_V \partial\mathbf{u}^{\mathrm{T}} \mathbf{f}\, \mathrm{d}V + \int_S \partial\mathbf{u}^{\mathrm{T}} \mathbf{t}\, \mathrm{d}S + \partial\mathbf{q}^{\mathrm{T}}\, \overline{\mathbf{Q}}, \tag{FE\_6}$$

where $\partial\mathbf{u}$ are virtual displacements, $\partial\boldsymbol{\varepsilon}$ are virtual strains, $\boldsymbol{\sigma}$ are engineering stresses, $\mathbf{f}$ are volumetric forces, $\mathbf{t}$ are traction forces, $\partial\mathbf{q}$ are virtual displacements of nodes, $\overline{\mathbf{Q}}$ are generalized external forces acting at the nodes.

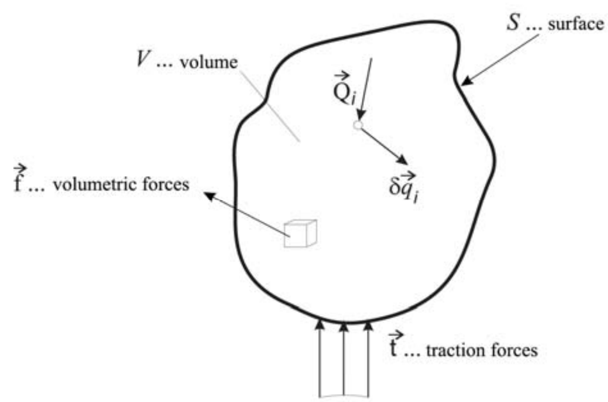


Fig. FE_1 ... Acting forces

Quantities V and S denote the volume and the surface of the body in the reference configuration. The integration process is carried out in reference (un-deformed) configuration, in agreement with accepted assumptions about small displacements and strains. This is what the linear theory is based on.

Deformation variant of the finite element method is based on the idea of approximation of continuous displacements of individual particles (material points) by polynomial functions. The approximation can be expressed by the relation $\mathbf{u} = \mathbf{A}\mathbf{q}$, by which we understand

$$\mathbf{u}_{\text{exact}} = \begin{Bmatrix} u_x(x,y,z,t) \\ u_y(x,y,z,t) \\ u_z(x,y,z,t) \end{Bmatrix} \approx \mathbf{u}_{\text{approx}} = \mathbf{u} = \mathbf{u}(\mathbf{x},t) = \mathbf{A}(\mathbf{x})\,\mathbf{q}(t). \tag{FE\_7}$$

In continuum, the displacement field of the body is continuous in time and space. In FE approximation this field is approximated by a product of so-called shape functions, contained in $\mathbf{A(x)}$ matrix, and of displacements $\mathbf{q}$ of certain, a priory set, and points – called nodes. The shape functions are polynomial functions of space and the nodal displacement are generally functions of time. This way, we approximate the continuous system, which has the infinite number of particles, and thus the infinite number of degrees of freedom, by a discrete medium, with the finite number of elements, having the finite number of degrees of freedom.

To make this process unique, the matrix $\mathbf{A}$ has to be determined. How to do it will be shown in the following paragraphs. Also, the initially continuous strains have to be discretized. This will be secured by another operator, i.e. $\mathbf{B}$, called stress-displacement operator, which relates strains to displacements in nodes by $\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{q}$.

By the strain approximation, we understand

$$\boldsymbol{\varepsilon}_{\text{exact}} = \begin{Bmatrix} \varepsilon_{xx}(x,y,z,t) \\ \varepsilon_{yy}(x,y,z,t) \\ \varepsilon_{zz}(x,y,z,t) \\ \gamma_{xy}(x,y,z,t) \\ \gamma_{yz}(x,y,z,t) \\ \gamma_{zx}(x,y,z,t) \end{Bmatrix} \approx \boldsymbol{\varepsilon}_{\text{approx}} = \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{x},t) = \mathbf{B}(\mathbf{x})\,\mathbf{q}(t). \qquad \text{(FE\_8)}$$

The $\mathbf{A}$ and $\mathbf{B}$ operators will be derived in the following text. So far, we can state that the $\mathbf{B}$ operator will depend on $\mathbf{A}$ due to the existence of kinematic relations.

In linear cases, the kinematic relations are simplified as

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial\, {}^0x_j} + \frac{\partial u_j}{\partial\, {}^0x_i}\right) \approx \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \qquad \text{(FE\_9)}$$

The virtual displacements and strains depend on accepted approximations given by Eqs. (FE_7), (FE_8) , so

$$\partial\mathbf{u} = \mathbf{A}\,\partial\mathbf{q} + \partial\mathbf{A}\,\mathbf{q}, \quad \partial\boldsymbol{\varepsilon} = \mathbf{B}\,\partial\mathbf{q} + \partial\mathbf{B}\,\mathbf{q}. \qquad \text{(FE\_10)}$$

Since the operators A,B do not depend on displacements, the relations Eq. (FE_10) simplify to

$$\partial\mathbf{u} = \mathbf{A}\,\partial\mathbf{q}, \quad \partial\boldsymbol{\varepsilon} = \mathbf{B}\,\partial\mathbf{q}. \qquad \text{(FE\_11)}$$

The volumetric forces might represent inertia forces. Using d'Alembert's principle we can write

$$\mathbf{f} = -\rho\ddot{\mathbf{u}}. \qquad \text{(FE\_12)}$$

It is worth mentioning that in continuum mechanics the volumetric forces are defined as forces related to a unit of volume, so their dimensions are $\left[\mathrm{Nm}^{-3}\right]$. Double dots, superimposed on displacements, represent the second derivative with respect time – that is the particle acceleration. It is approximated by

$$\ddot{\mathbf{u}} = \mathbf{A}\ddot{\mathbf{q}}. \qquad \text{(FE\_13)}$$

We tacitly assume that the $\mathbf{A}$ operator is not a function of time. The principle of virtual work, formulated for a discretized body, can be obtained by substituting the above assumptions into Eq. (FE_6). Rearranging we get

$$\partial\mathbf{q}^{\mathrm{T}}\left[\int_V \mathbf{B}^{\mathrm{T}}\boldsymbol{\sigma}\,\mathrm{d}V + \int_V \rho\,\mathbf{A}^{\mathrm{T}}\mathbf{A}\,\mathrm{d}V\,\ddot{\mathbf{q}} - \int_S \mathbf{A}^{\mathrm{T}}\mathbf{t}\,\mathrm{d}S - \overline{\mathbf{Q}}\right] = 0. \qquad \text{(FE\_14)}$$

This equation has to be valid for any virtual displacement $\mathrm{d}\mathbf{q}$. To satisfy this condition, the contents of the bracket must be identically equal to zero. Rearranging we get

$$\rho\int_V \mathbf{A}^{\mathrm{T}}\mathbf{A}\,\mathrm{d}V\,\ddot{\mathbf{q}} = R - \int_V \mathbf{B}^{\mathrm{T}}\boldsymbol{\sigma}\,\mathrm{d}V, \qquad \text{(FE\_15)}$$

where the $\mathbf{R}$ vector covers contributions of both traction and point forces. The term appearing by acceleration is called mass matrix and is denoted $\mathbf{m}$. We assume that during the deformation process the mass is conserved, so the mass matrix is constant. The Eq. (FE_15) holds at any moment, that is at the beginning, at the time $t = 0$, so

$$^{0}\mathrm{C}: \quad \mathbf{m}\ {}^{0}\ddot{\mathbf{q}} = {}^{0}\mathbf{R} - \int \mathbf{B}^{\mathrm{T}}\ {}^{0}\boldsymbol{\sigma}\ \mathrm{d}V \qquad \text{(FE\_16)}$$

as well as at a generic time $t > 0$, so

$$^{t}\mathrm{C}: \quad \mathbf{m}\ {}^{t}\ddot{\mathbf{q}} = {}^{t}\mathbf{R} - \int \mathbf{B}^{\mathrm{T}}\ {}^{t}\boldsymbol{\sigma}\ \mathrm{d}V. \qquad \text{(FE\_17)}$$

In linear cases, the changes in geometry are neglected. If, furthermore, the linear relation between stress and strain is considered, then we can write

$$^{t}\boldsymbol{\sigma} = \mathbf{C}\ {}^{t}\boldsymbol{\varepsilon} = \mathbf{C}\mathbf{B}\ {}^{t}\mathbf{q}. \qquad \text{(FE\_18)}$$

The quantity $\mathbf{C}$ represents the symmetric matrix of elastic moduli. Substituting Eq. (FE_18) into Eq. (FE_17) we get

$$\mathbf{m}\ {}^{t}\ddot{\mathbf{q}} + \mathbf{k}\ {}^{t}\mathbf{q} = {}^{t}\mathbf{R}. \qquad \text{(FE\_19)}$$

The previous relation, valid for a generic time *t*, represents the discretized equation of motion for a generic element, where

$$\mathbf{m} = \rho\int_V \mathbf{A}^{\mathrm{T}}\mathbf{A}\,\mathrm{d}V \qquad \text{is the mass matrix and} \qquad \text{(FE\_20)}$$

$$\mathbf{k} = \int_V \mathbf{B}^{\mathrm{T}}\mathbf{C}\mathbf{B}\,\mathrm{d}V \qquad \text{is the stiffness matrix.} \qquad \text{(FE\_21)}$$

It is not difficult to add another term representing the viscous damping being proportional to the velocity

$$\mathbf{m}\ {}^{t}\ddot{\mathbf{q}} + \mathbf{d}\ {}^{t}\dot{\mathbf{q}} + \mathbf{k}\ {}^{t}\mathbf{q} = {}^{t}\mathbf{R} \qquad \text{(FE\_22)}$$

but a meaningful determination of material constants might pose a problem. Sometimes a so-called Rayleigh's approach, defining the damping matrix

$$\mathbf{d} = \alpha \mathbf{m} + \beta \mathbf{k} \tag{FE\_23}$$

with constants α, β, is employed. See [4].

If inertial effects could be neglected then the equation of motion (FE_19) becomes the equilibrium equation having the form

$$\mathbf{k}\, {}^t\mathbf{q} = {}^t\mathbf{R}\,. \tag{FE\_24}$$

In linear cases there is no need to keep the upper left-hand side index, denoting the configuration state, since the final result is obtained by a single computational step, i.e. by solving the system of algebraic equations, supplying the unknown displacements.

The integration of Eqs. (FE_14), (FE_15) and (FE_20), (FE_21) has to be carried out separately for each element. Then, the individual results (mass and stiffness matrices) have to be systematically assembled to represent the inertia and stiffness properties of the whole body. From now on, we will assume that the volume *V* belongs to a generic element as well as that the matrices **m**, **d** and **k**. They are called mass, damping and stiffness matrices respectively. Equations of motion for the whole body have formally the same form, only instead of the local matrices **m**, **d** and **k** we formally write **M**, **D,** and **K**, meaning the global mass, damping and stiffness matrices respectively.

The global matrices are obtained from local ones by so-called assembly process, based on an idea of so-called nodal compatibility, meaning that the displacements on element boundaries are continuous (going from one element to another) and that the nodal forces could be systematically assembled into vectors. The assembling will be described in the following text.

11.9. Determination of A, B operators

Let's start with Lagrangian and Hermitian elements that are frequently used in technical practice. They are based on the idea of Lagrangian or Hermitian polynomial interpolations. There are also other interpolation procedures. See [11].

Let's remind the Lagrangian interpolation procedure for a function of one variable. See [35], [36]. For a given function $y = f(x)$ defined in the interval $< a,b >$ one has to find a suitable approximation in the form of a polynomial function based on the knowledge of a few functional values within that interval.

Knowing *n* couples of values (x_i, y_i) within the interval $< a,b >$ it is possible to find a polynomial function of the $(n-1)^{\text{th}}$ degree passing through all the known *n* points. We can write

$$y_{\text{approx}} = c_1 + c_2 x + c_3 x^2 + \cdots + c_n x^{n-1} = \mathbf{Uc}\,, \qquad \text{(FE\_25)}$$

where

$$\mathbf{U} = \left[1\, x\, x^2 \cdots x^{n-1}\right] \text{ and } \mathbf{c} = \{c_1\, c_2 c_3 \cdots c_n\}^{\mathrm{T}}\,. \qquad \text{(FE\_26)}$$

Values x_i define the locations of points in which the approximation is being provided. They are often called nodes. Values y_i represent the function values at nodes. The **U** is the so-called *matrix of approximation functions*; the **c** vector contains unknown coefficients of the polynomial approximation. Substituting all *n* couples of (x_i, y_i) into the previous relation we get

$$\mathbf{y} = \mathbf{Sc}\,, \qquad \text{(FE\_27)}$$

where

$$\mathbf{c} = \{c_1\, c_2 c_3 \cdots c_n\}^{\mathrm{T}}\,, \qquad \text{(FE\_28)}$$

$$\mathbf{y} = \{y_1\, y_2\, y_3 \cdots y_n\}^{\mathrm{T}}\,, \qquad \text{(FE\_29)}$$

and

$$\mathbf{S} = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ 1 & \vdots & \vdots & \cdots & \vdots \\ 1 & \vdots & \vdots & \cdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}. \qquad \text{(FE\_30)}$$

The unknown coefficients c_i are determined from the condition $\mathbf{c} = \mathbf{S}^{-1}\mathbf{y}$. Finally, the approximation function can be expressed in the form

$$y_{\text{approx}} = \mathbf{Uc} = \mathbf{US}^{-1}\mathbf{y} = \mathbf{Ay}\,, \qquad \text{(FE\_31)}$$

where the **A** – matrix being the product of the approximation-function matrix **U** and the inverse of **S** matrix – containing not functions but pure numbers only – is called the shape function matrix. As stated above for *n* couples of points we get the Lagrangian polynomial of the $(n-1)^{\text{th}}$ degree.

They are the nodal displacements that play the role of unknown function values in the deformation variant of the finite element method. Then, the shape function matrix **A** secures the approximation of element displacements, based on displacements at nodes, while the **B**, secures the strain-displacement approximation. This will be shown in detail in the following text.

The Hermitean interpolation approach requires to deal not only with the values of functions at the nodes but also with their derivatives. For more details see [35].

11.10. Material non-linearity only

This case is based on assumptions of small displacements, rotations, and small strains. The only non-linearity entering the game is the non-linear constitutive relation. The usual procedure is based on replacing the actual non-linear stress-strain dependence by a series of relations which are linear by parts. This way, the engineering stress in a new configuration is

$$ {}^{t+\Delta t}\boldsymbol{\sigma} = {}^{t}\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma}, \qquad \text{(FE\_32)} $$

where the stress increment $\Delta\boldsymbol{\sigma}$ is expressed as a linear function of strain increment in the form

$$ \Delta\boldsymbol{\sigma} = {}_{t}\mathbf{C}\Delta\boldsymbol{\varepsilon}, \qquad \text{(FE\_33)} $$

where ${}^{t}\mathbf{C}$ is a tangential value to the $\mathbf{C} = \mathbf{f}(\boldsymbol{\varepsilon})$ function at the ${}^{t}\mathbf{C}$ configuration. Introducing so-called *nodal displacement increment* we get

$$ \Delta\mathbf{q} = {}^{t+\Delta t}\mathbf{q} - {}^{t}\mathbf{q} \qquad \text{(FE\_34)} $$

and then the *strain increment* could be expressed in the form

$$ \Delta\boldsymbol{\varepsilon} = \mathbf{B}\Delta\mathbf{q}\,. \qquad \text{(FE\_35)} $$

Substituting into (FE_17) and rearranging we get

$$ \mathbf{m}\,{}^{t+\Delta t}\ddot{\mathbf{q}} + {}_{t}\mathbf{k}\,\Delta\mathbf{q} = {}^{t+\Delta t}\mathbf{R} - {}^{t}\mathbf{F}, \qquad \text{(FE\_36)} $$

where

$$ {}_{t}\mathbf{k} = \int_{V} \mathbf{B}^{\mathrm{T}}\mathbf{C}\,\mathbf{B}\,\mathrm{d}V \qquad \text{(FE\_37)} $$

is the *tangential stiffness matrix* and

$$ {}^{t}\mathbf{F} = \int_{V} \mathbf{B}^{\mathrm{T}}\,{}^{t}\boldsymbol{\sigma}\,\mathrm{d}V \qquad \text{(FE\_38)} $$

is the *vector of internal forces at nodes* and the vector ${}^{t+\Delta t}\mathbf{R}$ is the loading at time $t+\Delta t$.

The integration goes across the non-deformed volume. Using a suitable assumption for expressing the acceleration at the time $t+\Delta t$ as a function of displacement at time *t*, one can, using Eq. (FE_36), express $\Delta\mathbf{q}$ as the first estimation of displacement increment that must, however, be refined in a subsequent iteration process. The condition required for a successful iteration process is based on achieving the equilibrium of internal and external forces. The details could be found in [7], [12a].

11.11. Material and geometrical nonlinearity

By this we understand cases where there is a nonlinear relation between stress and strain; and large displacements and large strains are taken into account. One of the possible approaches is based on employing Green-Lagrange strain tensor with the second Piola-Kirchhoff stress tensor. For more details see [4], [7].

11.12. Finite element method in mechanics of deformable bodies

Linear static problems are numerically treated by solving the system of algebraic equations $\mathbf{Kq} = \mathbf{Q}$ where $\mathbf{K}$ is the stiffness matrix, $\mathbf{Q}$ is the loading vector and $\mathbf{q}$ is the unknown vector of generalized displacements at nodes.

When nonlinear static problems are solved, the stiffness matrix is not constant – generally it is a function of unknown displacements, the system to be solved is $\mathbf{K}(\mathbf{q})\mathbf{q} = \mathbf{Q}$ and requires the iterative solvers.

Linear steady state vibration problems are numerically treated by solving the generalized eigenvalue problem, which is defined by $(\mathbf{K} - \lambda\mathbf{M})\bar{\mathbf{x}} = \mathbf{0}$, where $\mathbf{M}$ is the mass matrix and $\mathbf{K}$ is the stiffness matrix. The sought-after quantities are the eigenmodes $\bar{\mathbf{x}}$ and eigenvalues λ.

Generally, n pairs could be found for the system with n degrees of freedom. The eigenvalues are related to eigenfrequencies by the relation $\lambda_i = \Omega_i^2$.

Transient dynamical problems are treated by solving the system of ordinary differential equations, which – in linear cases without damping – has the form of ordinary differential equations, i.e. $\mathbf{M}\ddot{\mathbf{q}} + \mathbf{Kq} = \mathbf{R}$. The loading vector is a function of time. Explicit and implicit numerical procedures are used to obtain the solution. The solution consists of a series of displacements, velocities, accelerations, strains, and stresses for each time step. See [10], [13], [14], [20], [28], [31], [34], [42].

A general procedure for defining mass and stiffness matrices was illustrated in broad terms in the previous text. Now, we will show how these matrices – for a few simple elements – are derived in detail. We will use the standard approach based on so-called generalized coordinates. Later, we will show another approach which leads to so-called isoparametric elements. The former process could be summarized as follows.

Displacement approximation is secured by a polynomial function of coordinates in the form

$$\mathbf{u} = \mathbf{Uc}\,, \qquad \text{(FE\_39)}$$

where $\mathbf{u}$ is the column vector containing the displacements to be approximated and $\mathbf{U}$ is the matrix of approximation functions, containing the function terms appearing in the approximation polynomial. The column vector **c** contains the constants of the assumed polynomial function. Substituting nodal coordinates into Eq. (FE_39) we get

$$\mathbf{q} = \mathbf{S}\mathbf{c}, \qquad \text{(FE\_40)}$$

where **q** is a column vector of nodal displacements and **S** is the matrix containing numbers, i.e. nodal coordinates and their powers. For elements which are not geometrically deteriorated (we will explain this term soon) the **S** matrix is regular, could be inverted, and so

$$\mathbf{c} = \mathbf{S}^{-1}\mathbf{q}. \qquad \text{(FE\_41)}$$

Substituting Eq. (FE_41) into Eq. (FE_40) we get

$$\mathbf{u} = \mathbf{U}\mathbf{S}^{-1}\mathbf{q} = \mathbf{A}\mathbf{q}, \qquad \text{(FE\_42)}$$

where $\mathbf{A} = \mathbf{U}\mathbf{S}^{-1}$ is the *matrix defining shape functions*. Strain approximation could be generally expressed in the form $\boldsymbol{\varepsilon} = f(\mathbf{u})$, whose discretized form is

$$\boldsymbol{\varepsilon} = \mathbf{F}\mathbf{c}, \qquad \text{(FE\_43)}$$

where the **F** matrix contains the derivatives of functions appearing in **A**. Substituting Eq. (FE_41) into Eq. (FE_43) we get

$$\boldsymbol{\varepsilon} = \mathbf{F}\mathbf{S}^{-1}\mathbf{q} = \mathbf{B}\mathbf{q}, \qquad \text{(FE\_44)}$$

where **B** is the *strain-displacement operator*. Now, the stiffness and mass matrices could be expressed in the forms

$$\mathbf{k} = \int_V \mathbf{B}^{\mathrm{T}}\mathbf{C}\mathbf{B}\,\mathrm{d}V, \qquad \text{(FE\_45)}$$

$$\mathbf{m} = \int_V \mathbf{A}^{\mathrm{T}}\mathbf{A}\,\mathrm{d}V. \qquad \text{(FE\_46)}$$

In the following text, we will derive the mass and stiffness matrices for a few geometrically simple elements. Later, a procedure for assembling global matrices out of elementary ones will be presented.

11.13. Rod (bar) element

The element is schematically depicted in Fig. FE_2. This is the simplest element being used in the technical practice. It lives in an one-dimensional space defined by its axial (longitudinal) axis.

It can transmit only axial forces and knows nothing about the bending or torsion. With its neighbors is connected, at its boundary nodes.

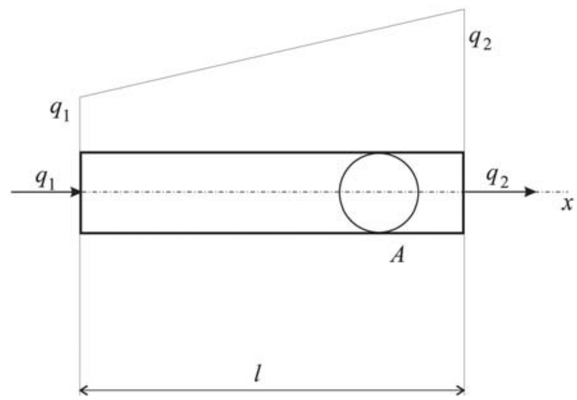


Fig. FE_2 ... Rod element with two dof's

It communicates with neighbors by means of two axial displacements q_1 and q_2, defined at boundary nodes.

These displacements are measured in the local coordinate system *x*. To see their distribution in space, they are plotted perpendicularly to their actual directions. The element is characterized by its length *l*, the density *ρ* and the Young's modulus *E*.

Now, we are looking for a suitable polynomial approximation $\mathbf{u} = \mathbf{Aq}$ for this element. Since we have only two free nodal displacement to play with, the only available choice is the polynomial of the first degree which has two unknown constants, ie. c_1, c_2.

In the local coordinate system we can write

$$u_{\text{approx}} = u = c_1 + c_2 x = [1 \; x] \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \mathbf{Uc}. \tag{FE\_47}$$

Denoting the nodal displacements

$$u|_{x=0} = q_1 \text{ and } u|_{x=l} = q_2 \tag{FE\_48}$$

and realizing that the assumed approximation should holds for the nodes as well

$$\mathbf{q} = \mathbf{Sc}, \tag{FE\_49}$$

where

$$\mathbf{q} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}, \quad \mathbf{c} = \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}. \tag{FE\_50}$$

Eliminating out of our considerations the negative or the zero length of the element (this way we exclude geometrically deteriorated element) then the **S** matrix is regular and could be inverted. From Eq. (FE_49) we can express c and substituting it into Eq. (FE_47) we get

$$\mathbf{u} = \mathbf{Uc} = \mathbf{US}^{-1}\mathbf{q} = \mathbf{Aq}\,. \tag{FE\_51}$$

Then, carrying out the above multiplication, we get the shape matrix in the form

$$A = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/l & 1/l \end{bmatrix} = \begin{bmatrix} 1 - x/l & x/l \end{bmatrix} = \begin{bmatrix} a_1(x) & a_2(x) \end{bmatrix}. \tag{FE\_52}$$

When deriving the strain-displacement matrix **B** from $\boldsymbol{\varepsilon} = \mathbf{Bq}$ one has to take into account the proper kinematic relations. In the case of one-dimensional deformations, applicable for this element, we can write

$$\boldsymbol{\varepsilon} = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(\mathbf{Aq}) = \frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix} 1 - x/l & x/l \end{bmatrix} = \begin{bmatrix} -1/l & 1/l \end{bmatrix}\mathbf{q}\,. \tag{FE\_53}$$

Thus, the strain-displacement matrix (operator) is

$$\mathbf{B} = \begin{bmatrix} -1/l & 1/l \end{bmatrix}. \tag{FE\_54}$$

Notice that the **B** operator does not depend on x variable – it this case it is constant. It is not generally so. This is due to the fact that the linear approximation of displacements was assumed. And the derivative of a linear function is constant, of course. From this follows that using this approximation we obtained the element which has the constant distribution of strains along its length. That's why it is sometimes called constant strain element. Now, we have all the ingredients necessary to derive the mass and stiffness matrices. Using Eq. (FE_20) we can express the mass matrix

$$m = \rho \int_V \mathbf{A}^{\mathrm{T}}\mathbf{A}\, dV = \rho A \int_0^l \mathbf{A}^{\mathrm{T}}\mathbf{A}\, dx = \frac{\rho A l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}. \tag{FE\_55}$$

This is so-called the *consistent mass matrix*, being derived consistently in agreement with so far presented rules.

There is another approach to the derivation of the mass matrix, which is based on its diagonalization. It this case there is a nice physical interpretation stemming from the idea of concentrating the continuously distributed mass into nodes.

What we get is so-call *lumped* or *diagonal mass matrix*

$$m = \frac{\rho A l}{6}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}. \tag{FE\_56}$$

Using Eq. (FE_21) we get the stiffness matrix

$$\mathbf{k} = \int_V \mathbf{B}^{\mathrm{T}} \mathbf{C} \mathbf{B} \, \mathrm{d}V = \int_0^l \begin{bmatrix} -1/l \\ 1/l \end{bmatrix} E \begin{bmatrix} -1/l & 1/l \end{bmatrix} A \mathrm{d}x = \frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{FE\_57}$$

We have taken for granted that there is a linear relation between the stress and strain, expressed by Eq. (FE_3), and the fact that in 1D case the matrix of elastic moduli simplifies to a scalar, i.e. to Young's modulus, so $\mathbf{C} = E$.

According to the generally accepted terminology, we say that an element having n independent nodal displacements has n degrees of freedom.

11.14. Planar beam element

Let's consider a planar beam element of prismatic cross-sectional area, with the shearing forces and the bending moments, depicted in Fig. FE_3.

In the first approach, we are neglecting axial forces. The element is characterized by the cross-sectional area A, bending stiffness EI, density ρ and the length l_0

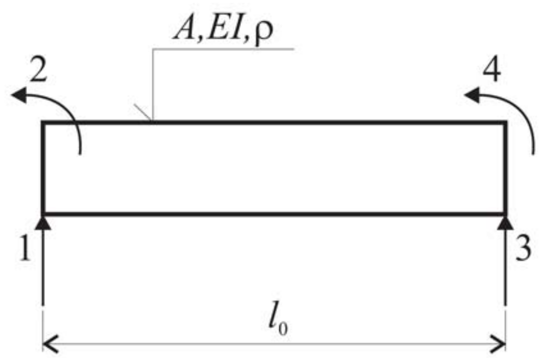


Fig. FE_3 ... Planar beam element with 4 dof's

Neglecting the axial forces, there are two displacements and two rotations at each node – altogether four degrees of freedom.

For more details see [30]. Let's approximate the vertical displacements by

$$\mathbf{u} = \{u_y(x)\} = \{c_1 + c_2 x + c_3 x^2 + c_4 x^3\} = \mathbf{U}\mathbf{c}, \tag{FE\_58}$$

where the x-coordinate goes along the longitudinal beam axial axis. Again, this approximation has to be valid at nodes as well, so

$$u_y(0) = q_1, \quad \frac{\mathrm{d}u_y(0)}{\mathrm{d}x} = q_2, \quad u_y(l_0) = q_3, \quad \frac{\mathrm{d}u_y(l_0)}{\mathrm{d}x} = q_4, \tag{FE\_59}$$

So,

$$\mathbf{q} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l_0 & l_0^2 & l_0^3 \\ 0 & 1 & 2l_0 & 3l_0^2 \end{bmatrix} = \mathbf{S}\mathbf{c}. \tag{FE\_60}$$

After inverting the **S** matrix

$$\mathbf{S}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/l_0^2 & -2/l_0 & 3/l_0^2 & -1/l_0 \\ 2/l_0^3 & 1/l_0^2 & -2/l_0^3 & 1/l_0^2 \end{bmatrix} \qquad \text{(FE\_61)}$$

we get

$$\mathbf{A} = \mathbf{U}\mathbf{S}^{-1} = \left[1 - 3x^2/l_0^2 + 2x^3/l_0^3 \quad x - 2x^2/l_0 + x^3/l_0^2 \quad 3x^2/l_0^2 - 2x^3/l_0^3 \quad -x^2/l_0 + x^3/l_o^2 \right].$$

... (FE_62)

In this case, the role of generalized strain is played by the beam curvature $\tilde{\varepsilon} = \mathrm{d}^2 u_y / \mathrm{d}x^2$ and also $u_y = \mathbf{A}\mathbf{q}$, and then $\tilde{\varepsilon} = (\mathrm{d}^2\mathbf{A}/\mathrm{d}x^2)\mathbf{q}$. Finally,

$$\mathbf{B} = \frac{\mathrm{d}^2\mathbf{A}}{\mathrm{d}x^2} = \left[-6/l_0^2 + 12x/l_0^3 \quad -4/l_0 + 6x/l_0^2 \quad 6/l_0^2 - 12x/l_0^3 \quad -2/l_0 + 6x/l_0^2 \right]. \qquad \text{(FE\_63)}$$

Using Eqs (FE_20), (FE_21) we get the mass and stiffness matrices

$$\mathbf{m} = \frac{\rho A l_0}{420} \begin{bmatrix} 156 & 22l_0 & 54 & -13l_0 \\ & 4l_0^2 & 13l_0 & -3l_0^2 \\ & & 156 & -22l_0 \\ \text{sym} & & & 4l_0^2 \end{bmatrix}, \qquad \text{(FE\_64)}$$

$$\mathbf{k} = \frac{2EJ}{l_0^3} \begin{bmatrix} 6 & 3l_0 & -6 & 3l_0 \\ & 2l_0^2 & -3l_0 & l_0^2 \\ & & 6 & -3l_0 \\ \text{sym} & & & 2l_0^2 \end{bmatrix}. \qquad \text{(FE\_65)}$$

11.15. Planar triangular element with 6 dof's

Let's consider a planar triangular element depicted in Fig. FE_4. In each node there is one displacement which can be decomposed into two components in directions of coordinate axes. Altogether the element has three displacement components in each direction – so six degrees of freedom. The triangular element was one the first element derived in history. A natural polynomial choice for such an element would be two linear functions of coordinates x and y, for each direction.

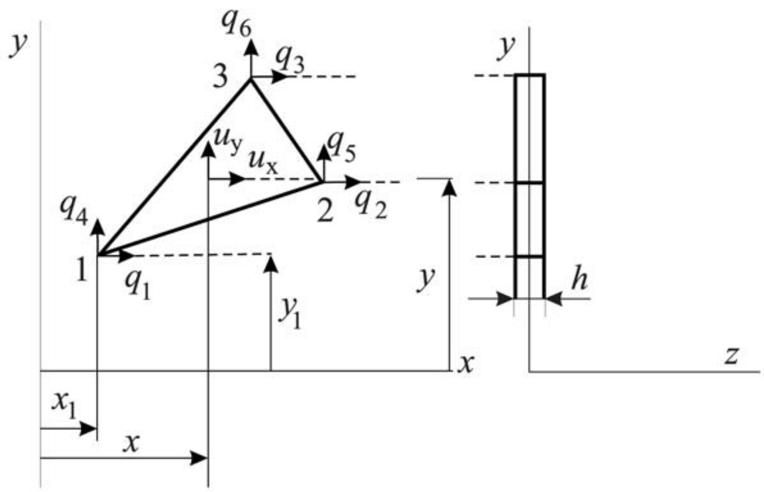


Fig. FE_4 ... Triangular element with 6 dof's

$$u = \begin{Bmatrix} u_x(x,y) \\ u_y(x,y) \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ c_6 \end{Bmatrix} = \mathbf{Uc}\,. \qquad \text{(FE\_66)}$$

Six unknown constants c_i of the approximation polynomial are found from the condition that Eq. (FE_66) must hold for all three nodes as well. Denoting the nodal coordinates by $x_i, y_i, i = 1,2,3$, then for all nodes we can write

$$\mathbf{q} = \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_6 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ c_6 \end{Bmatrix} = \begin{bmatrix} \tilde{\mathbf{S}} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{S}} \end{bmatrix} \mathbf{c}\,. \qquad \text{(FE\_67)}$$

The **S** matrix contains the coordinates of nodes. Unless the triangle degenerates into a line or into a single point then $\det(\mathbf{S}) \neq 0$ – matrix is regular – and one can write

$$\mathbf{c} = \mathbf{S}^{-1}\mathbf{q}\,. \qquad \text{(FE\_68)}$$

Substituting into Eq. (FE_66) we get $\mathbf{u} = \mathbf{U}\mathbf{S}^{-1}\mathbf{q}$. The analytical evaluation of $\mathbf{S}^{-1}$ is easy and gives

$$\mathbf{S}^{-1} = \begin{bmatrix} \tilde{\mathbf{S}}^{-1} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{S}}^{-1} \end{bmatrix}, \qquad \text{(FE\_69)}$$

where

$$S^{-1} = \frac{1}{\det \tilde{\mathbf{S}}} \begin{bmatrix} \mathbf{s}^{(1)} & \mathbf{s}^{(2)} & \mathbf{s}^{(3)} \end{bmatrix}, \qquad \text{(FE\_70)}$$

$$\mathbf{s}^{(1)} = \begin{Bmatrix} x_2 y_3 - x_3 y_2 \\ y_2 - y_3 \\ x_3 - x_2 \end{Bmatrix}, \quad \mathbf{s}^{(2)} = \begin{Bmatrix} x_3 y_1 - x_1 y_3 \\ y_3 - y_1 \\ x_1 - x_3 \end{Bmatrix}, \quad \mathbf{s}^{(3)} = \begin{Bmatrix} x_1 y_2 - x_2 y_1 \\ y_1 - y_2 \\ x_2 - x_1 \end{Bmatrix}.$$

$$\det \tilde{\mathbf{S}} = x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2). \qquad \text{(FE\_71)}$$

Denoting $\boldsymbol{\varphi}^{\mathrm{T}} = \{1 \; x \; y\}$ then the $\mathbf{A}$ matrix is

$$\mathbf{A} = \mathbf{U}\mathbf{S}^{-1} = \begin{bmatrix} \boldsymbol{\varphi}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\varphi}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{s}^{(1)} & \mathbf{s}^{(2)} & \mathbf{s}^{(3)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{s}^{(1)} & \mathbf{s}^{(2)} & \mathbf{s}^{(3)} \end{bmatrix} \frac{1}{\det \tilde{\mathbf{S}}} =$$
$$= \frac{1}{\det \tilde{\mathbf{S}}} \begin{bmatrix} a_1(x,y) & a_2(x,y) & a_3(x,y) & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1(x,y) & a_2(x,y) & a_3(x,y) \end{bmatrix}, \qquad \text{… (FE\_72)}$$

where $a_i(x,y) = \boldsymbol{\varphi}^{\mathrm{T}}\mathbf{s}^{(i)}$, $i = 1,2,3$ are linear functions of x, y. Expressing the displacements in one direction, say $u_x(x,y)$, then we can write

$$u_x(x,y) = (1/\det(\tilde{\mathbf{S}})(a_1 q_1 + a_2 q_2 + a_3 q_3).$$

Geometrical interpretation of this relation is depicted in Fig. FE_5. The plane ρ is defined by three nodal displacements q_1, q_2, q_3 (in one direction).

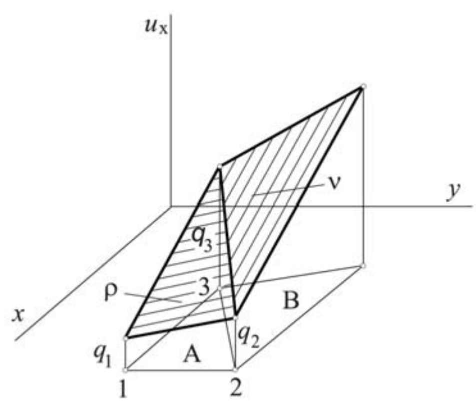


Fig. FE_5 ... Continuity of displacements

So far, one element, say A, was considered. For a neighbor element B we will get a similar plane, say ν, that must share with ρ plane two displacements – so the approximation of displacements has a common intersection line, which means that the approximation of displacements is continuous. The fact that both planes (approximations) have the same intersection line means that the approximated displacements are continuous, satisfying thus the so-called *compatibility conditions*. This cannot, however, be said of approximations of strains defined as derivatives of displacements. Since the approximations of displacements are defined by linear functions, their derivatives (approximations of strains) are constant within the element. This means that we have to live with fact that that approximation of strains (and stresses of course) will – for this kind of element – be discontinuous. There are strain jumps at element boundaries.

The strain displacement operator **B** depends on the type of stress state assumed. For the plane stress or the plane strain conditions, we start with Eqs. (FE_3) and (FE_66) obtaining

$$\frac{\partial u_x}{\partial x} = c_2, \quad \frac{\partial u_y}{\partial y} = c_6, \quad \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = c_3 + c_5, \tag{FE\_73}$$

which, written in the matrix form, gives

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial x \\ \partial u_y / \partial y \\ \partial u_x / \partial y + \partial u_y / \partial x \end{Bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{F}} \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ c_6 \end{Bmatrix} = \mathbf{Fc}. \tag{FE\_74}$$

We thus get $\boldsymbol{\varepsilon} = \mathbf{Fc}, \quad \mathbf{c} = \mathbf{S}^{-1}\mathbf{q}$ and finally

$$\boldsymbol{\varepsilon} = \mathbf{Bq}, \tag{FE\_75}$$

where $\mathbf{B} = \mathbf{FS}^{-1}$ so,

$$\mathbf{B} = \mathbf{FS}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{s}^{(1)} & \mathbf{s}^{(2)} & \mathbf{s}^{(3)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{s}^{(1)} & \mathbf{s}^{(2)} & \mathbf{s}^{(3)} \end{bmatrix} \frac{1}{\det \widetilde{\mathbf{S}}} =$$

$$= \frac{1}{\det \widetilde{\mathbf{S}}} \begin{bmatrix} y_2 - y_3 & y_3 - y_1 & y_1 - y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 & y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \end{bmatrix}. \qquad \text{...(FE\_76)}$$

In this case, the strain displacement matrix **B** contains nodal coordinates only – not functions. Alternatively, the **B** matrix could be obtained from **A** by evaluating the partial derivatives.

$$\mathbf{B}=\begin{bmatrix}\frac{\partial a_1}{\partial x} & \frac{\partial a_2}{\partial x} & \frac{\partial a_3}{\partial x} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{\partial a_1}{\partial y} & \frac{\partial a_2}{\partial y} & \frac{\partial a_3}{\partial y}\\ \frac{\partial a_1}{\partial y} & \frac{\partial a_2}{\partial y} & \frac{\partial a_3}{\partial y} & \frac{\partial a_1}{\partial x} & \frac{\partial a_2}{\partial x} & \frac{\partial a_3}{\partial x}\end{bmatrix}. \qquad \text{(FE\_77)}$$

The mass and stiffness matrices of this element could then be obtained by integrating relations (FE_20) and (FE_21), using $\mathrm{d}V = h\,\mathrm{d}x\,\mathrm{d}y$, where h is the element thickness, being considered constant.

11.16. Quadrilateral element with 8 dof's

The dimensions and node numbering are in Fig. FE_6.

The element lives in the plane, its thickness is h. The displacement approximation requires such a polynomial which has the same number of free polynomial constants as there is the numbers of nodal displacements in each direction. In this case, an incomplete polynomial of the second degree, having the form of a bilinear function – satisfying the spatial isotropy requirements, could be used. See [4], [11].

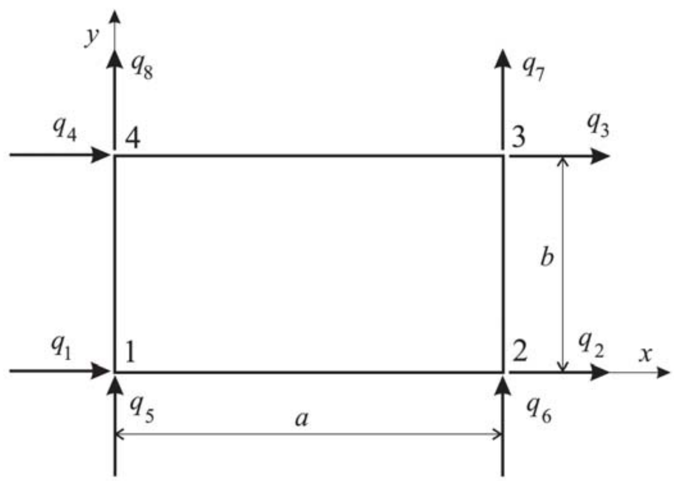


Fig. FE_6 ... Quadrilateral element with 8 dof's.

The displacement approximation could be expressed by

$$\mathbf{u} = \{u_x, u_y\}^{\mathrm{T}} = \mathbf{U}\mathbf{c}, \qquad \text{(FE\_78)}$$

where

$$U = \begin{bmatrix}\boldsymbol{\varphi}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}}\\ \mathbf{0}^{\mathrm{T}} & \boldsymbol{\varphi}^{\mathrm{T}}\end{bmatrix} \qquad \text{(FE\_79)}$$

and

$$\boldsymbol{\varphi}^{\mathrm{T}} = \{1 \quad x \quad y \quad xy\}, \quad \mathbf{0}^{\mathrm{T}} = \{0 \quad 0 \quad 0 \quad 0\}, \quad \mathbf{c}^{\mathrm{T}} = \{c_1 \quad \cdots \quad c_8\}.$$

Substituting the nodal coordinates into Eq. (FE_78) we get

$$\mathbf{q} = \mathbf{S}\mathbf{c} = \begin{bmatrix} \tilde{\mathbf{S}} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{S}} \end{bmatrix} \mathbf{c}, \quad \tilde{\mathbf{S}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & a & 0 & 0 \\ 1 & a & b & ab \\ 1 & 0 & b & 0 \end{bmatrix}, \qquad \text{(FE\_80)}$$

where $\mathbf{0}$ is 4x4 matrix full of zeros and $\mathbf{q}^{\mathrm{T}} = \{q_1 \dots q_8\}$ is a column vector of nodal displacements arranged with agreement of numbering shown in Fig. FE_6.

The analytical derivation of mass and stiffness matrices is provided by Matlab Symbolic Toolbox. See the program **symb_q4_mk**.

```
% symb_q4_mk
% mass and stiffness matrices of a rectangular elements
% for the plane stress
% a,b dimensions
% h thickness
% ro density
% mi Poisson ration
% E Young modulus
clear; format compact
% declaration of symbolic variables
syms fi x y s a b u h ro F B C mi Bt E p q;
fi = [1 x y x*y]; % approximation polynomial
zero = [0 0 0 0];
u = [fi zero; zero fi];   % matrix of approx. functions
S = [1 0 0 0 0 0 0 0; ... % matrix S
     1 a 0 0 0 0 0 0; ...
     1 a b a*b 0 0 0 0; ...
     1 0 b 0 0 0 0 0; ...
     0 0 0 0 1 0 0 0; ...
     0 0 0 0 1 a 0 0; ...
     0 0 0 0 1 a b a*b; ...
     0 0 0 0 1 0 b 0];
sinv = inv(S); % inversion of S matrix
aa = u*sinv;  % shape function matrix A
aat = aa.';   % transpose of A
ata = aat*aa; % integrand without constants
m1=int(ata,'y'); % integration with respect to y
mu=subs(m1,'y','b'); ml=subs(m1,'y','0'); % substitute limits
m2 = mu-ml; % subtract
m3=int(m2,'x'); % integration with respect x
mu=subs(m3,'x','a'); ml=subs(m3,'x','0'); % substitute limits
m4 = mu - ml; % subtract
m4 = ro*h*m4; % multiply by constants
const = 36/(a*b*h*ro); %
disp('mass matrix - multiplication by a*b*h*ro/36 is omitted')
m4 = const*m4
% stiffness matrix
% derivatives of approx. functions
dfix = diff(fi,x); dfiy = diff(fi,y);
% create F matrix
F = [dfix zero; ...
zero dfiy; ...
dfiy dfix];
% B matrix
B = F*sinv;
% transpose of B
Bt = B.';
% matrix of elastic constants for the plane stress
% with omitted constant ... constk
```

```
constk = E*h/(1-mi*mi);
C = [1 mi 0; ...
mi 1 0; ...
0 0 (1-mi)/2];
% integrand of the stiffness matrix
btcb = Bt*C*B;
% integration with respect to x and y variables within a,b
% thickness h is constant
k1 = int(btcb,'y'); % integrace podle y
ku = subs(k1,'y','b'); kl = subs(k1,'y','0'); % substitute limits
k2 = ku - kl; % subtract
k3 = int(k2,'x'); % integration with respect x
ku = subs(k3,'x','a'); kl = subs(k3,'x','0'); % substitute limits
k = ku - kl; % subtract
k = constk*k;
k = subs(k, {'a/b', 'b/a'}, {'p', 'q'});
k = subs(k, {'1/3/b*a', '1/6/a*b'}, {'p/3', 'q/6'});
k = subs(k, {'1/6/b*a', '1/6/a*b'}, {'p/6', 'q/6'});
constk = (1-mi^2)/(E*h); k = constk*k; simplify(k);
k = -24*k; k = simplify(k);
disp(' ')
disp('stiffness matrix')
disp('multiplication constant E*h/(24*(mi^2 - 1)) is omitted ')
disp('the first part k(1:8,1:4)'); disp(k(1:8,1:4))
disp('the second part k(1:8,5:8)'); disp(k(1:8,5:8))
% end of symb_q4_mk
```

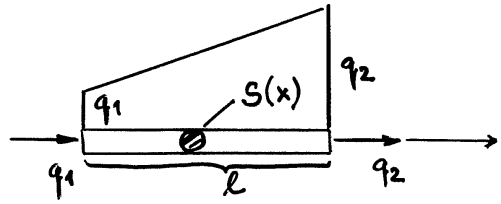
The program gives

```
mass matrix - multiplication by a*b*h*ro/36 is omitted

m4 = [ 4, 2, 1, 2, 0, 0, 0, 0]
     [ 2, 4, 2, 1, 0, 0, 0, 0]
     [ 1, 2, 4, 2, 0, 0, 0, 0]
     [ 2, 1, 2, 4, 0, 0, 0, 0]
     [ 0, 0, 0, 0, 4, 2, 1, 2]
     [ 0, 0, 0, 0, 2, 4, 2, 1]
     [ 0, 0, 0, 0, 1, 2, 4, 2]
     [ 0, 0, 0, 0, 2, 1, 2, 4]

stiffness matrix
multiplication constant E*h/(24*(mi^2 - 1)) is omitted

the first part k(1:8,1:4)
[ -8*q-4*p+4*p*mi, 8*q-2*p+2*p*mi, 4*q+2*p-2*p*mi, -4*q+4*p-4*p*mi]
[ 8*q-2*p+2*p*mi, -8*q-4*p+4*p*mi, -4*q+4*p-4*p*mi, 4*q+2*p-2*p*mi]
[ 4*q+2*p-2*p*mi, -4*q+4*p-4*p*mi, -8*q-4*p+4*p*mi, 8*q-2*p+2*p*mi]
[ -4*q+4*p-4*p*mi, 4*q+2*p-2*p*mi, 8*q-2*p+2*p*mi, -8*q-4*p+4*p*mi]
[ -3*mi-3, 9*mi-3, 3*mi+3, -9*mi+3]
[ -9*mi+3, 3*mi+3, 9*mi-3, -3*mi-3]
[ 3*mi+3, -9*mi+3, -3*mi-3, 9*mi-3]
[ 9*mi-3, -3*mi-3, -9*mi+3, 3*mi+3]

the second part k(1:8,5:8)
[ -3*mi-3, -9*mi+3, 3*mi+3, 9*mi-3]
[ 9*mi-3, 3*mi+3, -9*mi+3, -3*mi-3]
[ 3*mi+3, 9*mi-3, -3*mi-3, -9*mi+3]
[ -9*mi+3, -3*mi-3, 9*mi-3, 3*mi+3]
[ -8*p-4*q+4*q*mi, -4*p+4*q-4*q*mi, 4*p+2*q-2*q*mi, 8*p-2*q+2*q*mi]
[ -4*p+4*q-4*q*mi, -8*p-4*q+4*q*mi, 8*p-2*q+2*q*mi, 4*p+2*q-2*q*mi]
[ 4*p+2*q-2*q*mi, 8*p-2*q+2*q*mi, -8*p-4*q+4*q*mi, -4*p+4*q-4*q*mi]
[ 8*p-2*q+2*q*mi, 4*p+2*q-2*q*mi, -4*p+4*q-4*q*mi, -8*p-4*q+4*q*mi]
```

Four basic finite elements were derived – just to feel the flavor of the method. There are hundreds of elements available in technical practice. For more information see [4], [11].

11.17. Coordinate transformation

11.17.1. Rod element (2 dof's) in plane



depicted in Fig. FE_7, was derived in the local coordinate system. Let's denote it by $\bar{x}$, while the global coordinate system will denoted by x, y.

Fig. FE_7 ... Local displacements of the rod element

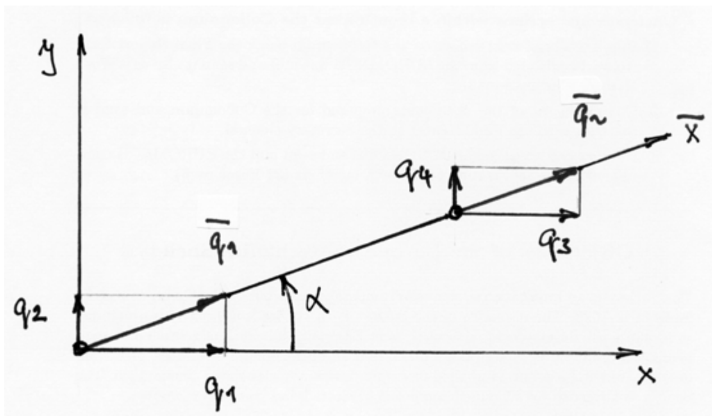


Fig. FE_8 ... Global displacements of the rod element

Both coordinate systems are shown in Fig. FE_8. The angle α is measured from the global to the local system. The relations between nodal displacements in local and global coordinate systems are

$$\bar{q}_1 = q_1 \cos\alpha + q_2 \sin\alpha , \qquad \text{(FE\_81a)}$$
$$\bar{q}_2 = q_3 \cos\alpha + q_4 \sin\alpha . \qquad \text{(FE\_81b)}$$

Written in matrix form, we have

$$\begin{Bmatrix} \bar{q}_1 \\ \bar{q}_2 \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}, \qquad \text{(FE\_82)}$$

or $\bar{\mathbf{q}} = \mathbf{T}\mathbf{q}$, (FE_83)

where the transformation matrix is

$$\mathbf{T} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \end{bmatrix}. \qquad \text{(FE\_84)}$$

Forces are vector quantities of the same kind, so

$$\overline{\mathbf{P}} = \mathbf{T}\mathbf{P}. \qquad \text{(FE\_85)}$$

The inverse relation to (FE_85) could be obtained by means of the following reasoning. The work done by external forces on virtual displacements should be independent of the coordinate system in which the displacements and forces are expressed, so

$$\delta\mathbf{q}^{\mathrm{T}}\mathbf{P} = \delta\overline{\mathbf{q}}^{\mathrm{T}}\overline{\mathbf{P}}, \qquad \text{where} \quad \overline{\mathbf{q}} = \mathbf{T}\mathbf{q}; \qquad \delta\overline{\mathbf{q}} = \mathbf{T}\delta\mathbf{q}; \qquad \delta\overline{\mathbf{q}}^{\mathrm{T}} = \delta\mathbf{q}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}, \qquad \text{(FE\_86)}$$

and

$$\delta\mathbf{q}^{\mathrm{T}}\mathbf{P} = \delta\mathbf{q}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}\overline{\mathbf{P}} \qquad \Rightarrow \delta\mathbf{q}^{\mathrm{T}}(\mathbf{P} - \mathbf{T}^{\mathrm{T}}\overline{\mathbf{P}}) = 0.$$

This relation must hold for any virtual displacement – this requires that the contents of the bracket must be identically equal to zero. So the inverse relation to (FE_85) is

$$\mathbf{P} = \mathbf{T}^{\mathrm{T}}\overline{\mathbf{P}}. \qquad \text{(FE\_87)}$$

Forces in the local coordinate system are proportional to displacements expressed in the same system. Briefly, we call them local forces and local displacements. They are related by the *local stiffness matrix* $\overline{\mathbf{k}}$. So, in the local system, we have

$$\overline{\mathbf{P}} = \overline{\mathbf{k}}\overline{\mathbf{q}}. \qquad \text{(FE\_88)}$$

This relation must be valid in the global coordinate system, as well. So,

$$\mathbf{P} = \mathbf{k}\mathbf{q}. \qquad \text{(FE\_89)}$$

Starting with Eq. (FE_87)

$\mathbf{P} = \mathbf{T}^{\mathrm{T}}\overline{\mathbf{P}}$ and substituting for $\overline{\mathbf{P}} = \overline{\mathbf{k}}\overline{\mathbf{q}}$ we get $\mathbf{P} = \mathbf{T}^{\mathrm{T}}\overline{\mathbf{k}}\overline{\mathbf{q}}$. Substituting then $\overline{\mathbf{q}} = \mathbf{T}\mathbf{q}$ we finally get

$$\mathbf{P} = \mathbf{T}^{\mathrm{T}}\overline{\mathbf{k}}\mathbf{T}\mathbf{q}, \qquad \text{(FE\_90)}$$

which might be rewritten into

$$\mathbf{P} = \mathbf{k}\mathbf{q}, \qquad \text{(FE\_91)}$$

where we have defined the stiffness matrix of rod element in global coordinates by

$$\mathbf{k} = \mathbf{T}^{\mathrm{T}}\overline{\mathbf{k}}\mathbf{T}\,. \tag{FE\_92}$$

Let's recall

$$\overline{\mathbf{k}} = \frac{ES}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \tag{FE\_93}$$

where E is Young's modulus, S is the cross-sectional area and l is the element length. In practice the stiffness matrices and their transformations are routinely are evaluated and the FE user is not burdened with processing details. Here, just for pleasure and for pedagogical reasons, we will explicitly do it step by step. See the following short program using the Matlab symbolic features

```
% mpp_stiffness_matrix_tranf_rod
clear
syms k k_bar T sin cos
k_bar = [1 -1; -1 1];
T = [cos sin 0 0 ; 0 0 cos sin];
k = T.'*k_bar*T;
pretty(k)
```

The output is

```
[    cos^2,  cos*sin,   -cos^2, -cos*sin]
[  cos*sin,    sin^2, -cos*sin,   -sin^2]
[   -cos^2, -cos*sin,    cos^2,  cos*sin]
[ -cos*sin,   -sin^2,  cos*sin,    sin^2]
```

Notice that the multiplicative constant $\frac{ES}{l}$ was intentionally omitted. So the stiffness matrix of the rod element, expressed in the global coordinate system, written in full, is

$$\mathbf{k} = \frac{ES}{l}\begin{bmatrix} \cos^2\alpha & \cos\alpha\sin\alpha & -\cos^2\alpha & -\cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha & -\cos\alpha\sin\alpha & -\sin^2\alpha \\ -\cos^2\alpha & -\cos\alpha\sin\alpha & \cos^2\alpha & \cos\alpha\sin\alpha \\ -\cos\alpha\sin\alpha & -\sin^2\alpha & \cos\alpha\sin\alpha & \sin^2\alpha \end{bmatrix}. \tag{FE\_94}$$

11.17.2. The planar beam element with 6 dof's

Degrees of freedom of a planar beam with 6 degrees of freedom are schematically depicted in Fig. FE_9.

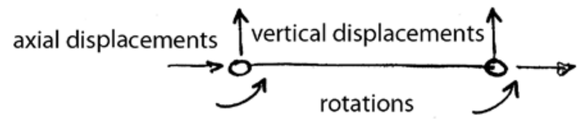


Fig. Fig. FE_9 ... Local dof's (displacements and rotations) of the beam element

The transformation of generalized displacements from the local to the global coordinate system is depicted in Fig. FE_10. The local coordinate system is $\bar{x}, \bar{y}$. The global coordinate system is x, y.

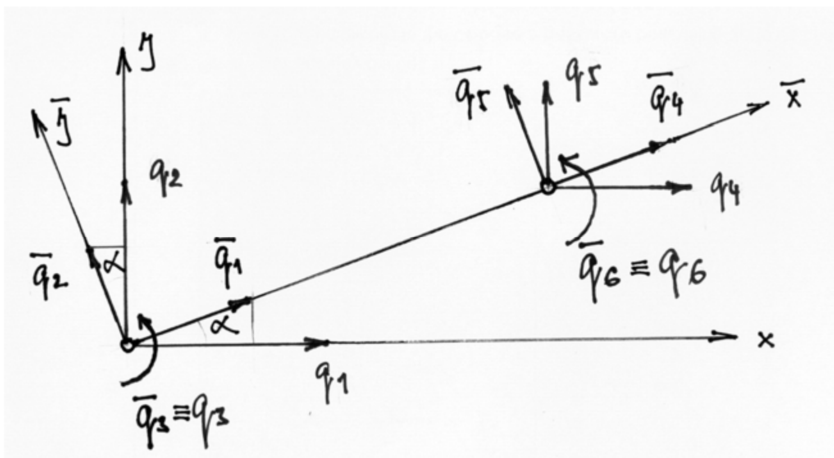


Fig. FE_10 ... Local and global dof's of the beam element

The angle α is measured from the global to local axes.

Observing Fig. FE_10 one can conclude that the relations between the generalized coordinates in global and local system are

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & 0 & 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{q}_1 \\ \bar{q}_2 \\ \bar{q}_3 \\ \bar{q}_4 \\ \bar{q}_5 \\ \bar{q}_6 \end{Bmatrix}. \tag{FE\_95}$$

In matrix form

$$\mathbf{q} = \mathbf{R}\bar{\mathbf{q}}, \tag{FE\_96}$$

where

$$\mathbf{R} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & 0 & 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{FE\_97}$$

The $\mathbf{R}$ matrix is orthogonal (which means that inverse is equal to its transposition), so

$$\overline{\mathbf{q}} = \mathbf{R}^{\mathrm{T}}\mathbf{q} = \mathbf{T}\mathbf{q}. \tag{FE\_98}$$

Analogically for generalized forces

$$\overline{\mathbf{P}} = \mathbf{T}\mathbf{P}. \tag{FE\_99}$$

We formally introduced

$$\mathbf{T} = \mathbf{R}^{\mathrm{T}} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{FE\_100}$$

In the local coordinate system, we have

$$\overline{\mathbf{P}} = \overline{\mathbf{k}}\overline{\mathbf{q}}. \tag{FE\_101}$$

Substituting for $\overline{\mathbf{P}}$ and $\overline{\mathbf{q}}$ we get

$$\mathbf{T}\mathbf{P} = \overline{\mathbf{k}}\mathbf{T}\mathbf{q} \tag{FE\_102}$$

and after multiplication by $\mathbf{T}^{\mathrm{T}}$ from the left, we obtain

$$\mathbf{P} = \mathbf{T}^{\mathrm{T}}\overline{\mathbf{k}}\mathbf{T}\mathbf{q} = \mathbf{k}\mathbf{q}, \tag{FE\_103}$$

where the stiffness matrix of the beam element in the global coordinate system is

$$\mathbf{k} = \mathbf{T}^{\mathrm{T}}\overline{\mathbf{k}}\mathbf{T}\mathbf{q}. \tag{FE\_105}$$

The stiffness matrix of the beam element in the local coordinate system, see [30], is

$$
\bar{\mathbf{k}} = \begin{bmatrix}
\frac{ES}{l} & 0 & 0 & -\frac{ES}{l} & 0 & 0 \\
0 & \frac{12EJ}{l^3} & \frac{6EJ}{l^2} & 0 & -\frac{12EJ}{l^3} & \frac{6EJ}{l^2} \\
0 & \frac{6EJ}{l^2} & \frac{4EJ}{l} & 0 & -\frac{6EJ}{l^2} & \frac{2EJ}{l} \\
-\frac{ES}{l} & 0 & 0 & \frac{ES}{l} & 0 & 0 \\
0 & -\frac{12EJ}{l^3} & -\frac{6EJ}{l^2} & 0 & \frac{12EJ}{l^3} & -\frac{6EJ}{l^2} \\
0 & \frac{6EJ}{l^2} & \frac{2EJ}{l} & 0 & -\frac{6EJ}{l^2} & \frac{4EJ}{l}
\end{bmatrix}. \qquad \text{(FE\_106)}
$$

The transformation of the relation (FE_106) is left to the reader.

11.18. Assembling

So far, the mass and stiffness matrices have been derived in the so-called *local coordinate system*. We call them the local matrices expressed in the local coordinate system. Usually, the displacements, forces and the matrices themselves have to be recalculated into another coordinate system, which is uniquely defined for all the elements. Such a system is called the *global coordinate system* and the mass and stiffness matrices are then called the local matrices expressed in the global coordinate system.

For more details see [4], [11], [30].

After having expressed the mass and stiffness matrices of all the elements in the global coordinate system, it is necessary to find a systematic way, how to assemble them into so-called global matrices, which would then represent the inertia and stiffness properties of the whole system.

The assembling process is based on the topology and compatibility considerations. The topology means that we know who is the neighbor of whom, while the compatibility means that the continuity of displacements (in their approximation forms) has to be satisfied. In this paragraph, the variables with the hat, say $\hat{\mathbf{q}}$, will denote the *local variables in the local coordinate system*, the variables with the tilde, say $\tilde{\mathbf{q}}$, will denote the *local variables in the global coordinate system*.

And the variables without any upper accent, say $\mathbf{q}$, will indicate the *global variables in the global coordinate system*.

Let's show it on an example.

Denoting by the upper right-hand side index the element number, then the equation of motion of the *i*-th element in the global coordinate system is

$$\tilde{\mathbf{m}}^i \ddot{\tilde{\mathbf{q}}}^i + \tilde{\mathbf{k}}^i \tilde{\mathbf{q}}^i = \tilde{\mathbf{P}}^i . \qquad \text{(FE\_107)}$$

For the whole system created by *n* elements, we can – simply but rather non-efficiently – assemble a single equation in the form

$$\mathbf{m}_c \ddot{\tilde{\mathbf{q}}} + \mathbf{k}_c \tilde{\mathbf{q}} = \tilde{\mathbf{P}} , \qquad \text{(FE\_108)}$$

where

$$\tilde{\mathbf{P}} = \left\{ \tilde{\mathbf{P}}^1 \quad \tilde{\mathbf{P}}^2 \quad \cdots \quad \tilde{\mathbf{P}}^n \right\}^{\mathrm{T}}, \quad \tilde{\mathbf{q}} = \left\{ \tilde{\mathbf{q}}^1 \quad \tilde{\mathbf{q}}^2 \quad \cdots \quad \tilde{\mathbf{q}}^n \right\}^{\mathrm{T}}, \qquad \text{(FE\_109)}$$

$$\mathbf{m}_c = \begin{bmatrix} \mathbf{m}^1 & & & \\ & \mathbf{m}^2 & & \\ & & \ddots & \\ & & & \mathbf{m}^n \end{bmatrix}, \quad \mathbf{k}_c = \begin{bmatrix} \mathbf{k}^1 & & & \\ & \mathbf{k}^2 & & \\ & & \ddots & \\ & & & \mathbf{k}^n \end{bmatrix}. \qquad \text{(FE\_110)}$$

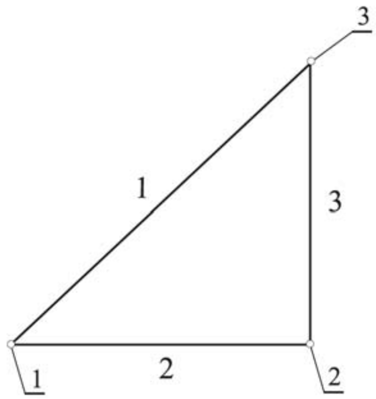
The vector of local nodal displacements of all elements in the global coordinate system $\tilde{\mathbf{q}}$ depends on the vector of global displacements of the system $\mathbf{q}$ by

$$\tilde{\mathbf{q}} = \mathbf{Z}\mathbf{q} , \qquad \text{(FE\_111)}$$

where $\mathbf{Z}$ is the so-called *incident matrix*. Each row of this matrix contains zeros with the exception a single '1' located at a place where the element of the vector $\tilde{\mathbf{q}}$ corresponds to the element of the vector $\mathbf{q}$. The process might be elucidated by the following example

Example

Let’s assemble the incident matrix $\mathbf{Z}$ for a ‚truss structure‘, formed by three rod elements, depicted in Fig. FE_11.



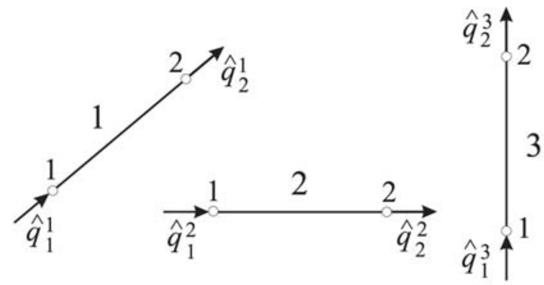


Fig. FE_11 ... Truss structure

Fig. FE_12 ... Nodal displacements of individual elements in local coordinates

In Fig. FE_12 to Fig. FE_14 we follow the transformation from the nodal displacements of elements in the local coordinate system $\hat{\mathbf{q}}$, through the nodal displacements of elements in the global coordinate system $\tilde{\mathbf{q}}$, to the displacements of the structure expressed in the global coordinate system, i.e. $\mathbf{q}$. In this case, the relation (FE_111) has the form

11.46
$$\begin{Bmatrix} \tilde{q}_1^1 \\ \tilde{q}_2^1 \\ \tilde{q}_3^1 \\ \tilde{q}_4^1 \\ \tilde{q}_1^2 \\ \tilde{q}_2^2 \\ \tilde{q}_3^2 \\ \tilde{q}_4^2 \\ \tilde{q}_1^3 \\ \tilde{q}_2^3 \\ \tilde{q}_3^3 \\ \tilde{q}_4^3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} = \begin{bmatrix} \mathbf{Z}^1 \\ \mathbf{Z}^2 \\ \mathbf{Z}^3 \end{bmatrix} \mathbf{q} \,. \qquad \text{(FE\_112)}$$

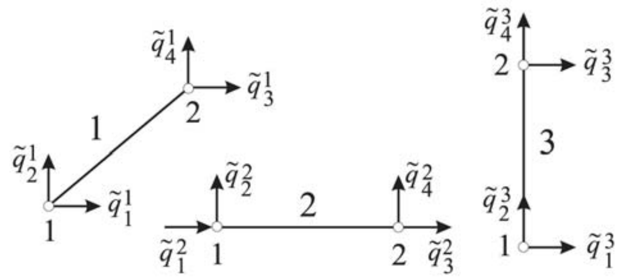


Fig. FE_13 ... Nodal displacements in global coordinates (individual elements)

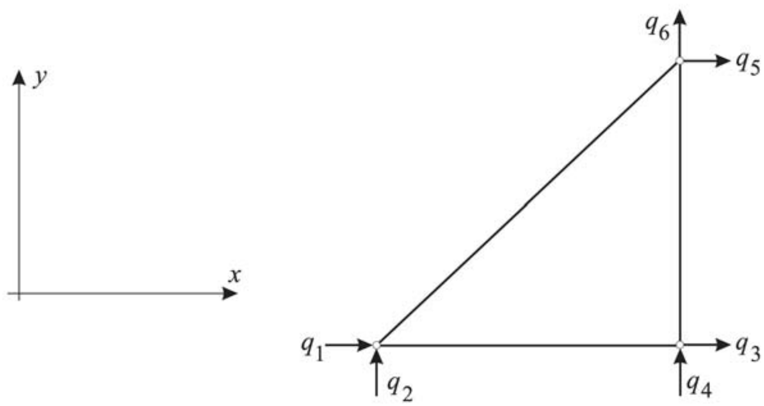


Fig. FE_14 ... Nodal displacements in global coordinates (truss structure)

Out of 12 nodal displacements corresponding to individual elements only 6 displacements are actually independent. This way the compatibility conditions are satisfied, meaning that the nodal displacements of neighboring elements are identical, say $\tilde{q}_1^2 = \tilde{q}_1^1$.

The principle of virtual work requires that the work done by inertia and internal forces must be equal to the work done external forces $\tilde{\mathbf{P}}$.

$$\partial \tilde{\mathbf{q}}^{\mathrm{T}} \mathbf{m}_c \ddot{\tilde{\mathbf{q}}} + \partial \mathbf{q}^{\mathrm{T}} \mathbf{k}_c \tilde{\mathbf{q}} = \partial \tilde{\mathbf{q}}^{\mathrm{T}} \tilde{\mathbf{P}} . \qquad \text{(FE\_113)}$$

Using $\partial \tilde{\mathbf{q}}^{\mathrm{T}} = \mathbf{Z} \partial \mathbf{q}, \quad \partial \ddot{\tilde{\mathbf{q}}}^{\mathrm{T}} = \mathbf{Z} \partial \ddot{\mathbf{q}}$ and Eq. (FE_111) and substituting into Eq. (FE_113) gives

$$\partial \mathbf{q}^{\mathrm{T}} \mathbf{Z}^{\mathrm{T}} \mathbf{m}_c \mathbf{Z} \ddot{\mathbf{q}} + \partial \mathbf{q}^{\mathrm{T}} \mathbf{Z}^{\mathrm{T}} \mathbf{k}_c \mathbf{Z} \mathbf{q} = \partial \mathbf{q}^{\mathrm{T}} \mathbf{Z}^{\mathrm{T}} \tilde{\mathbf{P}} . \qquad \text{(FE\_114)}$$

The last equation must hold for any virtual displacement $\partial \mathbf{q}$ which leads to

$$\mathbf{M} \ddot{\mathbf{q}} + \mathbf{K} \mathbf{q} = \mathbf{F} , \qquad \text{(FE\_115)}$$

where

$\mathbf{M} = \mathbf{Z}^{\mathrm{T}} \mathbf{m}_{\mathrm{c}} \mathbf{Z}$ is the global mass matrix, (FE_116)

$\mathbf{K} = \mathbf{Z}^{\mathrm{T}} \mathbf{k}_{\mathrm{c}} \mathbf{Z}$ is the global stiffness matrix, (FE_117)

$\mathbf{F} = \mathbf{Z}^{\mathrm{T}} \widetilde{\mathbf{P}}$ is the vector of external forces. (FE_118)

The elements of **F** vector are assembled the same way as those in the vector of displacements **q**. Generally, the **F** vector contains contributions of surface (traction) forces $\mathbf{P}^{(i)}$ acting on surfaces Ω_i, of initial stresses $\boldsymbol{\sigma}_0^{(i)}$ in volumes V_i, and the external forces $\mathbf{Q}^{(i)}$ acting in nodes – generally, it is a function of time.

$$\mathbf{F} = \sum_{i=1}^{n} \int_{\Omega_i} \mathbf{A}^{\mathrm{T}(i)} \mathbf{P}^{(i)} \,\mathrm{d}\Omega_i + \sum_{i=1}^{n} \int_{V_i} \mathbf{B}^{\mathrm{T}(i)} \boldsymbol{\sigma}_0^{(i)} \,\mathrm{d}V_i + \mathbf{Q}^{(i)} . \qquad \text{(FE\_119)}$$

In statics, the **F** vector does not depend on time and the inertia forces are neglected. Instead of Eq. (FE_115) we get

$$\mathbf{Kq} = \mathbf{F} . \qquad \text{(FE\_120)}$$

Example

In Fig. FE_11 the node numbers and element numbers of the considered truss structure are indicated. The stiffness matrix of the i-th element in local coordinates is

$$\mathbf{k}^{(i)} = \begin{bmatrix} k_{11}^{i} & k_{12}^{i} \\ k_{21}^{i} & k_{22}^{i} \end{bmatrix}, \quad i = 1,2,3. \qquad \text{(FE\_121)}$$

An equivalent matrix in global coordinates has the form

$$\widetilde{\mathbf{k}}^{i} = \begin{bmatrix} \widetilde{k}_{11}^{i} & \widetilde{k}_{12}^{i} & \widetilde{k}_{13}^{i} & \widetilde{k}_{14}^{i} \\ \widetilde{k}_{21}^{i} & \widetilde{k}_{22}^{i} & \widetilde{k}_{23}^{i} & \widetilde{k}_{24}^{i} \\ \widetilde{k}_{31}^{i} & \widetilde{k}_{32}^{i} & \widetilde{k}_{33}^{i} & \widetilde{k}_{34}^{i} \\ \widetilde{k}_{41}^{i} & \widetilde{k}_{42}^{i} & \widetilde{k}_{43}^{i} & \widetilde{k}_{44}^{i} \end{bmatrix}, \quad i = 1,2,3 . \qquad \text{(FE\_122)}$$

Let's create the table of so-called code numbers for this structure. The code numbers are actually the indices of global displacements, belonging to individual elements, listed in the same manner, i.e. from the local node 1 to the local node 2. The code numbers in our case are

element number	code numbers	
1	1 2 5 6	
2	1 2 3 4	(FE_123)
3	3 4 5 6	

Notice that the code numbers actually express the locations of 1's in the previously defined incident matrices Eq. (FE_115).Writing the code numbers around the first element matrix $\mathbf{k}^1$ we get

$$\begin{array}{ccccc} 1 & 2 & 5 & 6 & \\ \tilde{k}^1_{11} & \tilde{k}^1_{12} & \tilde{k}^1_{13} & \tilde{k}^1_{14} & 1 \\ \tilde{k}^1_{21} & \tilde{k}^1_{22} & \tilde{k}^1_{23} & \tilde{k}^1_{24} & 2 \\ \tilde{k}^1_{31} & \tilde{k}^1_{32} & \tilde{k}^1_{33} & \tilde{k}^1_{34} & 5 \\ \tilde{k}^1_{41} & \tilde{k}^1_{42} & \tilde{k}^1_{43} & \tilde{k}^1_{44} & 6 \end{array} \qquad \text{(FE\_124)}$$

Comparing with Eqs. (FE_116), (FE_117) we see that the code numbers have the meaning place-holders (pointers) indicating where the element of the local matrix is to be located in the global matrix. For example, the element k^1_{34} is to be located in the global matrix to the location defined by indices 5, 6. Graphically, the procedure is depicted in Fig. FE_15. The same way is followed when a mass matrix is to be assembled.

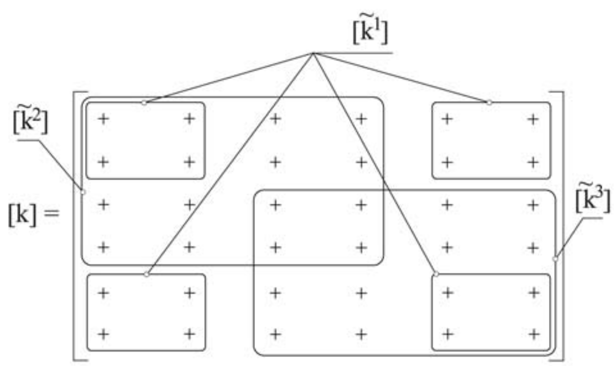


Fig. FE_15 ... Assembling the global matrix

Let's recall that the global matrix, assembled this way, corresponds to a mechanical system to which no boundary conditions have been prescribed yet – it floats freely in the space. Such a matrix cannot be inverted since it is singular – its determinant $\det \mathbf{K} = 0$.

11.19. Assembling algorithm

Let's have a system with `imax` (generalized) displacements, `kmax` elements, each element having `lmax` local (generalized) displacements, i.e. the local dof's – degrees of freedom. Furthermore, there exists a procedure `CODE(k,ic)`, which – when called – gives on its output the vector `ic(lmax)` containing the code numbers of the k-th element. The procedures `RIG(k,xke)` and

`MAS(k,xme)` generate the stiffness and mass matrices of the of the *k*-th element – i.e. the matrices `xke(lmax, lmax)` and `xme(lmax, lmax)`.

For simplicity, we assume that all the elements are of the same type and have the same number of dof's, i.e. `lmax`. The global matrices are `xk` a `xm`. In the old-fashioned Fortran, not exploiting the symmetry of matrices we are dealing with, we could write

```
C      Loop over elements
       DO 10 k = 1,kmax
C      Code numbers of the k-th element
       CALL CODE(k, ic)
C      Local matrices of the k-the element
       CALL RIG(k, xke)
       CALL MAS(k, xme)
C      Loop over elements of local matrices
       DO 20 k1 = 1,lmax
       DO 20 k2 = 1,lmax
C      Locations in global matrices and matrice themselves
       i1 = ic(k1)
       j1 = ic(k2)
       xk(i1,j1) = xke(k1,k2) + xk(i1,j1)
       xm(i1,j1) = xke(k1,k2) + xm(i1,j1)
20     CONTINUE
10     CONTINUE
```

In Matlab, where the vectors of pointers could appear at the index site of variables, the procedure is more elegant and substantially simpler.

```
% loop over elements
for k = 1:kmax
       ic = code(k);                % code numbers of the k-th element
       xke = rig(k); xme = mas(k);  % local matrices of the k-th element

       % assembly
       xm(ic,ic) = xm(ic,ic) + xme;
       xk(ic,ic) = xk(ic,ic) + xke;
end
```

11.20. Respecting boundary conditions

We have shown that the external (generalized) forces are related to (generalized) displacements by $\mathbf{Kq} = \mathbf{F}$. We already know how to assemble the global stiffness matrix, which, however, as it comes from the assembly process, is singular. It is due to the fact that so far the matrix knows nothing of boundary conditions and being singular cannot be inverted, not allowing to get displacements from $\mathbf{q} = \mathbf{K}^{-1}\mathbf{F}$. Evidently, a part of force components in $\mathbf{F}$ is due to reaction forces due to the way the body is constrained – attached to the fixed frame. Also, a part of displacements is already known, being dependent on the prescribed boundary conditions. To take these facts into account let's rearrange the 'equilibrium equation' in such a way that the known and unknown quantities are put apart. We can proceed as follows

$$\mathbf{F} = \mathbf{Kq}, \tag{FE\_125}$$

$$\begin{Bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{Bmatrix}, \tag{FE\_126}$$

where

$\mathbf{F}_1$	known external forces,
$\mathbf{F}_2$	unknown reactions,
$\mathbf{q}_1$	unknown displacements,
$\mathbf{q}_2$	prescribed displacements, representing boundary conditions.

Due to the symmetry of the stiffness matrix $\mathbf{K}$ it holds that $\mathbf{K}_{12} = \mathbf{K}_{21}^{\mathrm{T}}$.

From Eq. (FE_126) we get two matrix equations. From the first, solving the system of algebraic equations, we get

$$\mathbf{K}_{11}\mathbf{q}_1 = \mathbf{F}_1 - \mathbf{K}_{12}\mathbf{q}_2 \quad \Rightarrow \quad \mathbf{q}_1 . \tag{FE\_127}$$

Knowing $\mathbf{q}_1$, the second matrix equation leads to the evaluation of unknown reaction forces from

$$\mathbf{F}_2 = \mathbf{K}_{21}\mathbf{q}_1 + \mathbf{K}_{22}\mathbf{q}_2 . \tag{FE\_128}$$

If the system being solved is fixed to the frame in such a way that no mutual displacements between the body and the frame, are allowed, then we have $\mathbf{q}_2 = \mathbf{0}$ and the previous equations simplify to

$$\mathbf{K}_{11}\mathbf{q}_1 = \mathbf{F}_1 \quad \Rightarrow \mathbf{q}_1 , \tag{FE\_129}$$
$$\mathbf{F}_2 = \mathbf{K}_{21}\mathbf{q}_1 . \tag{FE\_130}$$

We could alternatively proceed by deleting the rows and columns from the Eq. (FE_126) which correspond to those dof's that represents the prescribed zero displacements, from the global 'unconstrained' $\mathbf{K}$ matrix.

Example

The boundary conditions could be prescribed in many ways. One of them is based on the idea of eliminating those degrees of freedom, which are a priory known that is to eliminate the generalized displacements which are, at the chosen supports, identically equal to zero. In other words, it requires deleting those rows and columns which correspond to the prescribed zero displacements. Formally, $\tilde{\mathbf{K}} \leftarrow \mathbf{K}, \quad \tilde{\mathbf{M}} \leftarrow \mathbf{M}$. This process is sometimes called the *static condensation*. The loading vector has to be submitted to this process as well, i.e. $\tilde{\mathbf{F}} \leftarrow \mathbf{F}$.

How it is done in Matlab.

```
% boundary conditions
% prescribe dof's where displacements,
% pointed to by elements of bc vector, are prescribed zero
 bc = [1 2 16 17];

% static condensation - delete corresponding rows and columns
 k_glob(bc,:) = [];
 k_glob(:,bc) = [];
 m_glob(bc,:) = [];
 m_glob(:,bc) = [];
 % delete corresponding items in the loading vector as well
 F(bc) = [];
```

In Matlab, the system of algebraic equations is solved by the backslash „\" operator. The unknown displacements are

```
displ = k_glob\F;
```

There is another way, how the stiffness matrix could be treated to recognize the boundary conditions and be thus regularized. Instead of rearranging the rows and columns, which is a computationally unpleasant operation, we could proceed in a following, approximate, way.

Imagine that in our stiffness matrix $\mathbf{K}$ we want to prescribe just one boundary condition, say $q_n = 0$. Let's replace the current diagonal element $k_{n,n}$ by a 'big' number, say m, which is at least 10^8 times larger than other element values appearing in the matrix. Now, the approximate inverse matrix of $\mathbf{K}$ could be obtained from the equation $\mathbf{K}\mathbf{K}^{-1} = \mathbf{I}$. Decomposing the matrix and writing the partial products in full we have

$$\begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{c}^{\mathrm{T}} & m \end{bmatrix} \begin{bmatrix} \mathbf{X} & \mathbf{y} \\ \mathbf{u}^{\mathrm{T}} & v \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix}. \tag{FE\_131}$$

Notice the different fonts used for the scalar values $m, v, 1$, the matrix values $\mathbf{A}, \mathbf{X}, \mathbf{I}, \mathbf{0}$, and for the vector values $\mathbf{b}, \mathbf{c}, \mathbf{y}$. It should be reminded that $\mathbf{b}$ represents the column vector, while $\mathbf{b}^{\mathrm{T}}$, its transpose, is the row vector.

From the previous equation, the following four equations could be written

$$\mathbf{A}\mathbf{X} + \mathbf{b}\mathbf{u}^{\mathrm{T}} = \mathbf{I}, \tag{FE\_132}$$
$$\mathbf{c}^{\mathrm{T}}\mathbf{X} + m\mathbf{u}^{\mathrm{T}} = \mathbf{0}^{\mathrm{T}}, \tag{FE\_133}$$
$$\mathbf{A}\mathbf{y} + v^{\mathrm{T}}\mathbf{b} = \mathbf{0}, \tag{FE\_134}$$
$$\mathbf{c}^{\mathrm{T}}\mathbf{y} + mv = 1. \tag{FE\_135}$$

From Eq. (FE_135)

$$v = \frac{1}{m}\left(1 - \mathbf{c}^{\mathrm{T}}\mathbf{y}\right), \tag{FE\_136}$$

can be substituted into Eq. (FE_134)

$$\mathbf{y} = \frac{1}{m}\left(\frac{1}{m}\mathbf{b}\mathbf{c}^{\mathrm{T}} - \mathbf{A}\right)^{-1}\mathbf{b}. \tag{FE\_137}$$

From Eq. (FE_133)

$$\mathbf{u}^{\mathrm{T}} = -\frac{1}{m}\mathbf{c}^{\mathrm{T}}\mathbf{X}. \tag{FE\_138}$$

Finally,

$$\mathbf{X} = \left(\mathbf{A} - \frac{1}{m}\mathbf{b}\mathbf{c}^{\mathrm{T}}\right)^{-1}. \tag{FE\_139}$$

Since m is very large, then $\frac{1}{m}$ is very small and thus

$$\mathbf{y} \to \mathbf{0}, \quad v \to 0, \quad \mathbf{X} \to \mathbf{A}^{-1} \quad \text{and} \quad \mathbf{u}^{\mathrm{T}} \to \mathbf{0}^{\mathrm{T}}. \tag{FE\_140}$$

So, the approximation of the inverse stiffness matrix is

$$\mathbf{K}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{bmatrix}. \tag{FE\_141}$$

Which is what we wanted to show.

12_RE. References

[1] Barrett, R., et al: Templates for the Solution of Linear Systems, SIAM, Philadelphia, 1994.

[2] Bathe, K.J.,Wilson, E.L.: Numerical Methods in Finite Element Analysis, Prentice-Hall, Englewood Cliffs, New Jersey, 1976.

[3] Bathe, K.J.: Finite element procedures in engineering analysis, Prentice-Hall, Englewood Cliffs, New York, 1982.

[4] Bathe, K.J.: Finite Element Procedures, Prentice-Hall, Englewood Cliffs, New Jersey, 1996.

[5] Beer, G., Watson, J.O.: Introduction to finite and boundary element methods for engineers, John Wiley, New York, 1991.

[6] Belytschko, T., Hughes, T.R.J.: Computational methods for transient analysis, North Holland, Amsterdam, 1986.

[7] Belytschko, T., Liu, W.K., Moran, B., Elkhodary, K.I.: *Nonlinear Finite Elements for Continua and Structures*, John Wiley & Sons, Ltd, 2014.

[8] Brebbia, C.A., Telles, J.C.F. a Wrobel, L.C.: Boundary element techniques, Springer Verlag, Berlin, 1984.

[9] Carnahan, B., Luther, H.A. a Wilkes, J.O.: Applied Numerical Methods, John Wiley, New York, 1969.

[10] Clough, R.W., Penzien, J.: Dynamics of Structures, McGraw-Hill, New York, 1993.

[11] Cook, R.D.: Concepts and applications of finite element method, John Wiley, New York, 1991.

[12a] Crisfield, M.A.: Non-linear finite element analysis of solids and structures, John Wiley, New York, 1991.

[12b] Davies, J.L.: Finite Difference Methods in Dynamics of Continuum Media, McMillan, New York, 1986.

[13] Dokainish, M.A., Subbaraj, K.: A survey of direct time integration methods in computational structural dynamics - I. Explicit methods, Computers and Structures, No. 6, pp. 1371–1386, 1989.

[14] Encyclopedia of computational mechanics, ISBN: 9781119003793, John Wiley & Sons, Ltd, 2018.

[15] Forsythe, G.E., Malcolm, M.A. a Moler, C.B.: Computer Methods for Mathematical Computations, Prentice-Hall, Englewood Cliffs, New York, 1977.

[16] Galileo Galilei: *Discorsi e Dimostrazioni matematiche*. In: Thomas Salusbury's Mathematical Collections and Translations, Tom II, p. 89, London, 1665.

[17] Gere, J. M. and Timoshenko, S.P.: *Mechanics of Materials,* PWS Publishing Company. 1997.

[18] Fung, K.Y.C., Tong, P., and Chen, X.: *Classical and Computational Solid Mechanics*, 2nd Edition, World Scientific Publishing, 2017, ISBN 978-981-4713-64-1.

[19] Hunter, S.C.: Mechanics of Continuous Media, Ellis Hornwood Ltd, Chichester, West Sussex, England, 1983.

[20] *Implementation of Nonlinear Mechanics in Finite Element Codes*, Edited by Okrouhlík, M. Institute of Thermomechanics, Prague, 1995.

[21] James, I., Gere, M.: *Mechanics of Materials*, Thomson Learning, Inc., 2004.

[22] Lanczos, C.: *The Variational Principles of Mechanics,* Dover Publications, Inc. 1970.

[23] Malvern, L.E.: Introduction to the mechanics of a continuous medium, Prentice-Hall, Englewood Cliffs, New York, 1969.
[24] Moler, C.: *Numerical Computing with MATLAB*, www.mathworks.com/moler
[25] *Numerical methods in computational mechanics*,
Edited by Okrouhlík, M. Institute of Thermomechanics, Prague 2008,
http://www.it.cas.cz/cs/ok
[26] Okrouhlík, M., Höschl, C.: One-dimensional finite elements in nonstationary elastodynamics, Acta Technica ČSAV, Vol. 30, No. 3, pp. 283–303, 1985.
[27] Okrouhlík, M., Höschl, C.: A contribution to the study of dispersive properties of one-dimensional Lagrangian and Hermitian elements, Computers and Structures, Vol. 49, No. 5, pp. 779–795, 1993.
[28] Okrouhlík, M.: Implementation of nonlinear continuum mechanics in finite element codes, Institute of Thermomechanics, Prague, 1995.
[29] Parlett, B.N.: The Symmetric Eigenvalue Problem, Prentice-Hall, Englewood Cliffs, New Jersey, 1978.
[30] Przemieniecki, J.S.: Theory of matrix structural analysis, McGraw-Hill, New York, 1968.
[31] Rao, S.S.: Mechanical Vibrations, Addison Wesley, Reading, Massachusetts, 1995.
[32] Sochi, T.: *Introduction to Tensor Calculus*, Department of Physics & Astronomy, University College London, Gower Street, London, WC1E 6BT, t.sochi@ucl.ac.uk, May 25, 2016.
[33] Strang, G., Fix, G.J.: An analysis of the finite element method, Prentice-Hall, Englewood Cliffs, N. J., 1973.
[34] Stejskal, V., Dehombreux, P., Eiber, A., Gupta, R., Okrouhlík, M.: *Mechanics with Matlab*, April 2001, http://www.it.cas.cz/cs/ok
[35] Strang, G.: Linear Algebra and its Applications, Academic Press, New York, 1976.
[36] Strang, G.: Introduction to Applied Mathematics, Introduction to Applied Mathematics, Welleseley - Cambridge Press, Welleseley, MA, U.S.A., 1986.
[37] *Le Système international d'unités. The International System of Units*. Bureau International des poids et mesures. Organisation intergouvernementale de la Convention du Mètre, 2006.
[38] Timoshenko, S.: *History of Strength of Materials*, Dover Publications, Inc., New York, ISBN 0-486-61187-6, 1983.
[39] Timoshenko, S. *Strength of Materials*, 3rd edition. Krieger Publishing Company, ISBN 0-88275-420-3, 1976.
[40] Todhunter, I., Pearson, K.: *A History of the Theory of Elasticity and of the Strength of Materials,* Vol. I and II., Dover Publications, 1960.
[41] Wasow, W.R., Forsythe, G.E.: Finite Difference Methods for Partial Differential Equations, John Wiley, New York, 1960.
[42] Zienkiewicz, O.C., Taylor, R.L.: The finite element method, McGraw-Hill, London, 1991.

Title	MECHANICS OF DEFORMABLE BODIES
Author	prof. Ing. Miloslav Okrouhlík, CSc.
Intended for	students of Bachelor Study Program Mechanical Engineering
Published by	Technical University of Liberec, Studentská 1402/2, Liberec
Approved by	Rectorate of TU Liberec on 18th December 2018, Ref. No RE 66/18
Published on	December 2018
Number of pages	229
Issue	1st
Publication number	55-066-18

ISBN 978-80-7494-453-6