TECHNICAL UNIVERSITY OF LIBEREC
Faculty of Mechanical Engineering

Prof. Ing. Miloslav Okrouhlík, CSc.

MECHANICS OF RIGID BODIES

2017

Reviewer: doc. Ing. Petr Hora, CSc.

ISBN 978-80-7494-394-2

Mechanics of rigid bodies – scope

I. Introductory part

1. Introduction
2. Foreword
3. Background for scalars, vectors and matrices
4. Background for statics, kinematics and dynamics

S. Statics

1. Introduction to statics
2. Forces, moments, torque
3. Principle of transmissibility
4. Equilibrium
5. Equivalence
6. Degrees of freedom
7. Constraints and free body diagram
8. Classification of constraints
9. Friction
10. Rolling resistance
11. Principle of virtual work
12. Internal forces
13. Centre of gravity, centre of mass, and static moment
14. References

K. Kinematics

1. Introduction to kinematics
2. Motion of particles
3. Rotary and translatory motion of bodies
4. Acceleration components appearing in a non-inertial frame of reference
5. Generic motion of bodies in two-dimensional space
6. References

D. Dynamics

1. Introduction to dynamics
2. Dynamics of a particle subjected to a straight line motion
3. Dynamics of a particle subjected to a motion along a curve
4. Dynamics of a particle subjected to a circular motion in plane
5. Newton's and d'Alembert's formulations of equations of motion
6. Vibrations
7. Moments of inertia and deviatoric moments
8. Dynamics of rigid bodies
    - Translatory motion
    - Rotary motion
        - Planar rotary motion
        - Spatial rotary motion about an axis
    - Planar general motion
    - Summary to dynamics of rigid bodies
9. References

Introductory part

Scope

1. Introduction
2. Foreword
3. Background for scalars, vectors, and matrices
4. Background for statics, kinematics, and dynamics

I1. Introduction

The presented text represents a background for undergraduate students attending one-semester course dedicated to mechanics of rigid bodies.

The course is based on classical deterministic Newtonian mechanics in which space and time coordinates are completely independent. It is assumed that the rigid, i.e. non-deformable, bodies have masses that are independent of their speeds, that bodies move with velocities that are negligible with respect to the speed of light, and furthermore that we can accept the notion of an inertial system – that is the system which is at rest or which moves with constant velocity with respect to the 'fixed stars'. Also, non-deterministic traps of quantum mechanics are avoided.

The course, divided into three parts, is subsequently devoted to

- Statics – analysis of forces acting on bodies – time variable is not considered.
- Kinematics – displacements, velocities, accelerations – no forces are considered.
- Dynamics – analysis of motions of bodies in time and space.

This course is a prerequisite to series of future lectures devoted to mechanics of deformable bodies which will mainly deal with

- Elastic deformations characterized by the fact that the relation between stress and strain, i.e. $\sigma = f(\varepsilon)$, is linear.
- Non-elastic deformations – no permanent deformations occur. The relation $\sigma = f(\varepsilon)$ is non-linear, but no hysteresis occurs.
- Non-elastic deformations – with permanent deformations. The relation $\sigma = f(\varepsilon)$ is non-linear, but there is a distinct hysteresis.

Another series of courses devoted to a broader subject of computational mechanics is prepared and will be available soon. Its intended scope is as follows

Computational Mechanics

- Continuum mechanics.
- Computer science.
- Numerical analysis.

It is assumed that students have the ability to routinely evaluate standard mathematical functions, and have the elementary knowledge of vector calculus, matrix analysis, differential and integral calculus. The above mentioned items constitute a sort of engineering craftsmanship.

The practical engineering result is required to be a number, a series of numbers and/or graphs based on which the thorough analysis and the rational engineering and managerial decisions are made. That's why a reader (= future engineer) should be able to enter and manipulate lists and arrays of numbers and to write short programs – for this purpose the Matlab is employed.

The text tries to explain the basic principles of mechanics of rigid bodies by detailed analysis of many worked-out examples. The enclosed short programs are intended to be read, played with and the obtained results should be thought about at length and in depth. Since it is only a one-semester course, many advanced items of analytical mechanics are omitted.

The course might be of interest to people intending to deal with commercial finite element packages, where a proper understanding of terminology and of basics of mechanical principles is a must.

The author can't resist to provide a few pieces of wisdom and to suggest the readers that the main goal to be achieved when studying mechanical engineering is to see things in proper relations, to be able to distinguish what is important and what could be neglected. One has to realize that the ability to find pieces of information somewhere on internet addresses does not establish the knowledge itself. Important are the relations between the pieces of information. And last but not least, the fundaments of understanding of mathematics and physics are required.

I2. Foreword

I2.1. Modeling

The computational mechanics, of which this course is an introductory part, generally aims to the modeling of large and non-trivial tasks in physics and in engineering practice. One has to emphasize that the proper understanding of the treated problem and the appropriate choice of the physical, mechanical, as well as numerical models, are crucial for the successful solution of tasks in question. To fully succeed, one should furthermore master algorithms of numerical analysis and to command the basics of computer science, that is programming, programming languages, operating systems, etc.

The model, as we understand it in physics and in mechanical engineering, is a purposeful simplification of an actual phenomenon in Mother Nature. It is created with the intention to predict – to describe what would be the behavior of the modeled phenomenon under the accepted simplifications. After that, one has to compare the model behavior with that of the modeled phenomenon. The assessment of model reliability and accuracy is usually based on properly conceived experiments. After the created model is thoroughly tested and satisfies our requirements on reliability and accuracy, then we do not need to perform the experiment.

So, the main goal of the modeling process is to predict the future without making excessive and repeated use of often difficult and rather expensive experiments. Of course, the experiments cannot be avoided since they are needed for validation of new models. The modeling that is properly validated is crucial for accepting meaningful decisions of engineering and/or managerial nature.

I2.2. Doubts

The results obtained by theoretical, numerical and experimental approaches in computational solid continuum mechanics are correlated and compared with intentions to ascertain which of them are 'truer' or closer to 'reality'. This, however, invokes many questions.

- How is truth related to consistency and validity of theoretical, numerical and experimental models we are inventing and employing?
- What is the role of threshold in physics, engineering, computation and in an experiment?
- How the basic quantities, as time, force, stress, etc. are defined? Do we properly understand them?
- What is the role of singularity in mathematics, physics and in engineering?

Answers to above questions are difficult to found and lead naturally to profound doubts. These difficulties, however, do not preclude our positive attitude to problem-solving. On the contrary, the presented text should persuade the reader to believe that the role of doubts in our understanding of Mother Nature plays a positive role.

I2.3. Truth

When trying to answer the question what is a true approach to modeling processes in physics and engineering we have to start inquiring about the notion of Truth.

Thomas Aquinas (1225 – 1274) claimed that the truth is an agreement of reality with perception. Today, however, the perceived reality depends on observation tools being used. For example, the results of observation obtained by the magnifying glass with those of an electron microscope are quite different.

Immanuel Kant (1724 – 1804) asked for a clear distinction between the 'true reality' and 'perceived reality'. Kant argues that *in principle it is impossible to observe and study the world without disturbing it*. His ideas are very close to those of Heisenberg principle of uncertainty.

As mentioned above, the model is a purposefully simplified concept of a studied phenomenon invented with the intention to predict – what would happen if … Accepted assumptions (simplifications) consequently specify the validity limits of the model and in this respect, the model is neither true nor false. The model – regardless of being simple or complicated – is good, if it is approved by an appropriately conceived experiment.

When we, engineers, are modeling particular phenomena of Nature, the question of truth becomes irrelevant since the models we are designing with, checking and using, either work or do

not work to our satisfaction. It is an undeniable fact that the mechanical theories, principles, laws, and models, used in engineering practice, cannot be proclaimed true or false. They are either right or wrong. Furthermore, the 'right' theories might fail when applied out of the limits of their applicability. A few examples might illustrate the previous claims.

- 1D wave equation is not able to predict stress wave pattern in a 3D body, and still is internally consistent and not wrong.
- Bernoulli-Navier's slender beam theory 'fails' for thick beams.
- Newton's second law 'fails' for motion of bodies approaching the speed of light, and still, it represents a perfect tool for engineering mechanics, including the computations and perfect prediction of celestial trajectories.
- Einstein's theory of relativity 'fails' when applied to quantum microcosms.

So it is obvious that we rather strive for robust models with precisely specified limits of validity and not for philosophically defined categories of truth and falsehood. From it follows that it is the validity of models, theories, and laws that is of primary importance. How do we proceed?

- When trying to reveal the 'true' behavior of a mechanical system we are using an experiment.
- When trying to predict the 'true' behavior of a mechanical system we are accepting a certain theoretical model and then solve it analytically and/or numerically.

The trouble is that the physical laws (or the models based upon them) cannot – in the mathematical sense – be proved. We cannot, for example, prove Newton's second law. On the other hand, the Pythagorean Theorem can be proved rather easily.

And still, one intuitively feels that a theorem is yet a less heavy-artillery term than a law. The terms, as law, theory, hypothesis, theorem, are not uniquely defined. 'Words, words, words'[1].

To get rid of doubts we often claim that it is the experiment, which ultimately confirms the model in question. But experiments, as well as the subsequent numerical treatment of models describing the nature, have their observational thresholds. And sometimes, the computational threshold of computational analysis is narrower than those of an experiment. From this point of view, a particular experiment is a model of nature as well.

In our incessant quest for truth we might have another mental hindrance, namely the lack of precise definitions of certain mechanical quantities. It appears that definitions of conceptually defined quantities as force, stress, energy, etc are rather intuitive and often circular.

Other widely used terms as stress, energy, etc. may generate similar doubts and questions.

[1] LORD POLONIUS: What do you read, my lord? HAMLET: Words, words, words. From Hamlet. SCENE II. A room in the castle.

I2.4. Concluding our ideas about modeling we might say

Mechanical theories, principles, laws, and models, used in engineering practice, cannot be proclaimed true or false. They are either right (working to our satisfaction) or wrong. Regardless of being simple or complicated, they are 'right', if approved by an *appropriate* experiment (i.e. the experiment conceived in agreement with accepted assumptions of the theory). History reveals that wrong theories might appear, but not being confirmed by experiments, are quickly discarded as ether or phlogiston. Theories might be right only within the limits of their applicability. We cannot claim that a theory being proved by an experiment is right. The only thing we can safely state is that such a theory is not proved wrong.

Generally, a singularity appearing in a model always means a serious warning concerning the range of validity of that model. Usually, a more general model – having a wider scope of validity – is invented removing that singularity. Very often there is no need to discard the older and simpler model since it might perfectly work in the validity range for which it was conceived.

The modeling process primarily consists of understanding the investigated phenomenon, in its decomposition into basic physical 'items', in establishing causal relations – often in terms of differential equations, whose solutions have to be found.

In simple cases[2] analytical solutions in closed forms are available. However, even in these cases, the solution is based on many physical, geometrical and numerical approximations.

In most cases, however, we have to systematically rely on approximate approaches based on physical simplifications, spatial and temporal discretizations, on numerical methods, on their efficient implementations, and last but not least on computers.

I3. Background for scalars, vectors, and matrices

I3.1. Scalars

The quantities fully determined by their magnitudes are called scalars. Temperature, energy or density, denoted as T, E, ρ, are good examples. In the presented text they are printed in italics.

I3.2. Vectors

Vectors are quantities uniquely determined by their magnitudes and directions. Examples are displacement, velocity, acceleration, force, moment, etc. They are denoted by a bar or by an arrow as $\bar{v}$ or $\vec{v}$. Sometimes they are printed by bold characters as $\mathbf{v}$ for example. The magnitude of the vector $\vec{v}$ is denoted $|\vec{v}|$ or v. In literature the terms velocity and speed are often distinguished. The former is used for a vector quantity, i.e. $\vec{v}$, while the latter is reserved for its magnitude, i.e. $v = |\vec{v}|$.

[2] As the deflection of a thin beam in the theory of linear elasticity.

Vectors are invariant with respect to a coordinate system. The choice of coordinate system is arbitrary, but a particular choice may be advantageous.

Frequently, the position of the origin of the directed line is immaterial. In such a case two vectors are considered identical if they are of the same length and direction. These vectors are referred to as *free vectors*.

Often, it is convenient to associate the vector with a line along which it can freely move. Such a line is often called the *line of action*. These vectors are referred to as *bound vectors*.

Still, there are vectors associated with a fixed point. They are referred to as *position, location* or *radius vectors*.

Any non-zero vector in 3D space can be expressed as a linear combination of three arbitrary non-zero base vectors. The most frequent choice of base vectors in the right-handed rectangular Cartesian system is the set of three unit vectors $\vec{i}, \vec{j}, \vec{k}$ aligned with coordinate axes. See Fig. I01. So, a vector, say $\vec{a}$, can be expressed by means of its scalar components a_x, a_y, a_z by

$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} .$$

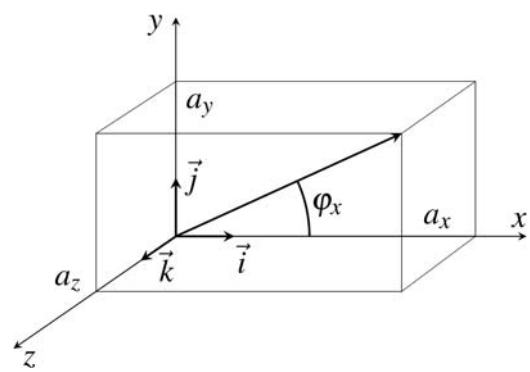


Fig. I01. Cartesian vector

Instead of naming the coordinate axes by x, y, z, we might alternatively denote them by $x_1\, x_2\, x_3$. Similarly, the base vectors, instead of $\vec{i}, \vec{j}, \vec{k}$, could be denoted by $\vec{e}_1, \vec{e}_2, \vec{e}_3$. This allows an efficient and elegant notation in the form of notation, i.e. $\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3 = \sum_{k=1}^{3} a_k\vec{e}_k = a_k\vec{e}_k$.

Notice, that behind the last equal sign of the previous formula, we have dropped the summation sign. This is in agreement with so-called summation convention (sometimes Einstein's rule) which states.

When an index appears twice in a term then that index is understood to take all the values in its range and the resulting term summed.

A few things, obvious from the above figure, are worth remembering.

Vector length: $a = |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} = (a_i a_i)^{\frac{1}{2}}$.

Direction cosines: $\cos\varphi_x = \frac{a_x}{|\vec{a}|}, \cos\varphi_y = \frac{a_y}{|\vec{a}|}, \cos\varphi_z = \frac{a_z}{|\vec{a}|}$.

Angle, say γ, between two vectors $\vec{a}, \vec{b}$ can be obtained from the relation $\cos\gamma = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$.

I3.3. Operations with vectors

I3.3.1. Addition, subtraction

Graphically, these operations are provided by so-called parallelogram law. See Fig. I02.

Numerically we proceed as follows

If $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$ and $\vec{b} = b_x\vec{i} + b_y\vec{j} + b_z\vec{k}$, then $\vec{a} \pm \vec{b} = (a_x \pm b_x)\vec{i} + (a_y \pm b_x)\vec{j} + (a_z \pm b_z)\vec{k}$.

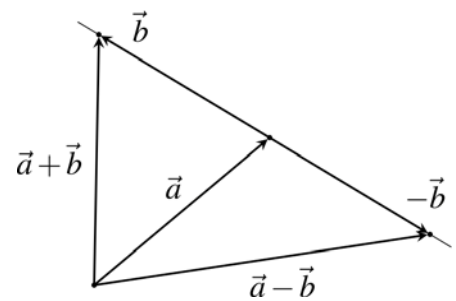


Fig. I02. Vector addition and substraction

I3.3.2. Multiplication

There are two kinds of vector multiplication defined.

a) *Dot multiplication* (also dot product, sometimes scalar product) of vectors, say $\vec{a}, \vec{b}$, yields a scalar quantity s. The dot serves as an operator of this operation. So, we write $s = \vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z = a_i b_i$.

If the angle between vectors $\vec{a}, \vec{b}$ is φ, then the dot product is $s = |\vec{a}||\vec{b}|\cos\varphi$. From it follows that the dot product of two perpendicular vectors is zero since $\cos\frac{\pi}{2} = 0$. If the former vector represents the force and the latter the displacement, then the physical meaning of the dot product is the mechanical work, or energy.

b) *Cross multiplication* (also vector product) of vectors, say $\vec{a}, \vec{b}$, gives a vector quantity $\vec{c}$. The operation is denoted by a cross sign, i.e. by operator $\times$. The resulting vector, say $\vec{c}$, is perpendicular to the plane formed by vectors $\vec{a}, \vec{b}$, so we write $\vec{c} = \vec{a}\times\vec{b}$.

The direction of the resulting vector is determined by so-called right-hand rule[3].

The vector product is defined by $\vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$.

The above determinant might be evaluated by means of the Sarus' rule which gives

$$\vec{c} = (a_y b_z - a_z b_y)\vec{i} + (a_z b_x - a_x b_z)\vec{j} + (a_x b_y - a_y b_x)\vec{k}.$$

The magnitude of this cross product is $|\vec{c}| = |\vec{a}||\vec{b}|\sin\varphi$ where the quantity φ is the angle between $\vec{a}$ and $\vec{b}$.

I3.3. Orthogonal transformation of a 2D vector

The same vector could be observed in two coordinate systems having a common origin but different orientations of axes as shown in Fig. I03.

One coordinate system has axes denoted by x, y, the other by x', y'. Even if the vector $\vec{a}$ is unique, its components in both coordinate systems are different.

The relation (also called the transformation) between components of the same vector in two different coordinate systems, is obtained by mere inspection of Fig. FI03, which gives

$$a_x = a_{x'} \cos\varphi - a_{y'} \sin\varphi,$$
$$a_y = a_{x'} \sin\varphi + a_{y'} \cos\varphi.$$

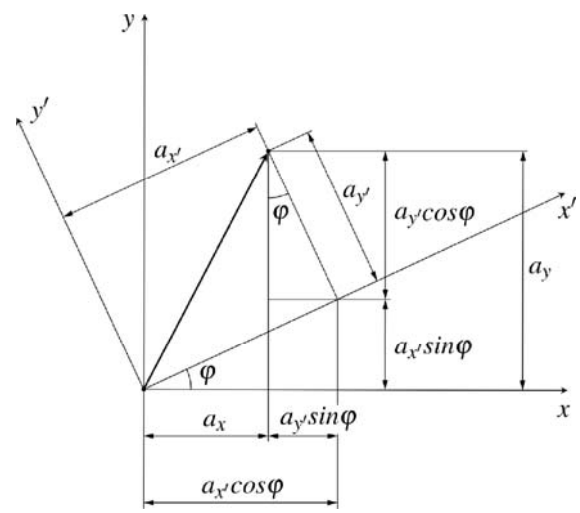


Fig. I03. Vector in two coordinate system

[3] If the thump points in the direction of the vector $\vec{a}$ – see Fig. S03 – and the index finger in the direction of the vector $\vec{b}$, then the middle finger points in the direction of the resulting vector $\vec{c}$.

In the matrix form, we have

$$\begin{Bmatrix} a_x \\ a_y \end{Bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{Bmatrix} a_{x'} \\ a_{y'} \end{Bmatrix}; \qquad \mathbf{a} = \mathbf{R}\mathbf{a}' .$$

In this case, the transformation matrix $\mathbf{R}$ represents the rotation process and is said to be orthogonal. For an orthogonal matrix its determinant $\det \mathbf{R} = \pm 1$ and its inverse is obtained by a mere transposition, i.e. $\mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}}$. So, the inverse transformation is defined by

$$\begin{Bmatrix} a_{x'} \\ a_{y'} \end{Bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \begin{Bmatrix} a_x \\ a_y \end{Bmatrix}; \qquad \mathbf{a}' = \mathbf{R}^{\mathrm{T}}\mathbf{a} .$$

I3.4. Orthogonal transformation of a 3D vector

Let the axes $O\,x_1, x_2, x_3$ and $O'\,x_1'\,x_2'\,x_3'$ represent two right handed Cartesian coordinate systems with a common origin at an arbitrary point $O \equiv O'$. If a symbol r_{ij} represents the cosine of an angle between *i*-th primed and *j*-th unprimed coordinate axes i.e. $r_{ij} = \cos(\text{angle between } x_i' x_j)$, then all the nine components can be arranged into a 3×3 matrix $\mathbf{R} = [r_{ij}]$, that is called the rotation matrix or the transformation matrix, or the matrix of direction cosines. Then, the transformation of a generic vector $\mathbf{a}$ is provided by same formulas as before, i.e. $\mathbf{a} = \mathbf{R}\mathbf{a}'$ and $\mathbf{a}' = \mathbf{R}^{\mathrm{T}}\mathbf{a}$.

I3.5. Matrices

The subject is fully treated in

- Okrouhlik, M.: Numerical methods in computational mechanics. Institute of Thermomechanics, Prague 2009, pp. 1 – 356, ISBN 978-80-87012-35-2. http://www.it.cas.cz/files/u1784/Num_methods_in_CM.pdf
- Stejskal, V., Dehombreux P., Eiber, A., Gupta, R., Okrouhlik, M.: Mechanics with Matlab, Electronic Textbook, ISBN 2-9600226-2-9, http://www.geniemeca.fpms.ac.be, Faculté Polytechnique de Mons, Belgium, April 2001

I3.6. Notation

Scalar variables are printed in lowercase or uppercase italics	as K, q, σ.
Matrix and vector variables are printed in bold fonts	as $\mathbf{K}, \mathbf{q}, \boldsymbol{\sigma}$.
Elements of matrices, are printed in italics, accompanied by indices	as K_{ij}, q_i, σ_i.
'True vectors' are printed with a bar or with an arrow or by bold fonts	as $\bar{v}, \vec{v}$ or $\mathbf{v}$.

Partial derivatives, as $\dfrac{\partial u_i}{\partial x_j}$ might be shortened to $u_{i,j}$.

I4. Background for statics, kinematics, and dynamics

The text is devoted to Newtonian mechanics which is valid for small velocities – small with respect to the speed of light. Under these conditions, the mass of a moving body is independent of its speed. In the theory of relativity, attributed to Albert Einstein, it is not so and it is assumed (and proved as well) that the current mass m depends on the rest mass m_0 by the relation

$$m = \frac{m_0}{\sqrt{1 - v^2 / c^2}} ,$$

where v is the current velocity of a moving body and c is the speed of light. It is obvious that as the velocity $v = |\vec{v}|$ approaches the speed of light c the denominator of the above formula goes to zero and thus the current mass in limit reaches infinity. So, in a limit we have

$$\lim_{v \to c} \frac{m_0}{\sqrt{1 - v^2 / c^2}} \to \infty .$$

From it follows that a body, having a non-zero mass, cannot reach the speed of light.

One should recall, however, that a photon always moves at the speed of light within a vacuum. But it supposedly has the zero rest mass.

To see things in proper relations

- Find the speed v needed for the current mass be doubled with respect to the rest mass. From the relation $2 = \frac{1}{\sqrt{1 - (v/c)^2}}$ we get $\frac{v}{c} = \frac{\sqrt{3}}{2} \cong 0.8660$. So, almost 87 % of the speed of light is required. Quite a lot – is it not?

- Using the above formula check how the rest mass $m_0 = 1\text{kg}$ is changed when the velocity of Earth (approximately 30 km/s) is taken into account. The result is $m = 1.000000005$ kg. Notice, that the relative difference is of the order of 10^{-9}, and thus the resulting error is negligible.

Both examples show that, when dealing with current mechanical engineering problems, we are on the save ground when considering the value of mass independent of velocity.

I4.1. Newton's laws

Newton describes force as the ability casing a body to accelerate. His three laws can be, for a mass point (particle), summarized as follows

1. **First law**: If there is no net force on a particle, then its velocity is constant. The particle is either at rest (if its velocity is equal to zero), or it moves with constant speed in a single direction.
2. **Second law**: The rate of change of linear momentum $\mathbf{p} = m\mathbf{v}$ of a particle of mass m is equal to the acting force **F**, i.e., d**p**/dt = **F**.
3. **Third law**: When a first body exerts a force $\mathbf{F}_1$ on a second body, the second body simultaneously exerts a force $\mathbf{F}_2 = -\mathbf{F}_1$ on the first body. This means that $\mathbf{F}_1$ and $\mathbf{F}_2$ are equal in magnitude and opposite in directions.

Newton's first and second laws, as stated above, are valid only in an inertial frame of reference. That is in the frame (sometimes called system) which is either in rest or moves with a constant velocity along a straight line with respect to fixed stars or by other words is subjected to no acceleration. Even if such a system does not actually exist in the Universe, the notion of an inertial frame of reference is a useful and frequent approximation for many technical cases.

Take the Earth for example. It rotates and moves with acceleration along its orbit and still, with accuracy sufficient for many (not for all[4]) engineering cases, is a good approximation of the inertial system.

For the safe application of Newton's laws in non-inertial frames of references, so-called apparent inertia forces, in agreement with d'Alembert principle, have to be introduced.

Newton's second law, written for a particle of mass m, states that the time rate of linear momentum is proportional to the external force

$$\frac{\mathrm{d}(m\vec{v})}{\mathrm{d}t} = \vec{F} \Rightarrow \frac{\mathrm{d}m}{\mathrm{d}t}\vec{v} + \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}m = \vec{F} \ .$$

The product of $m\vec{v}$ is called the *momentum*. Sometimes, the *linear momentum*. If the mass does not change in time, i.e. $m = const$, then we have the classical high-school formula in the form

$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}m = \vec{F} \Rightarrow m\vec{a} = \vec{F}$, since the acceleration is a time derivative of velocity.

Another possible formulation

$\mathrm{d}(m\vec{v}) = \vec{P}\mathrm{d}t$ … states that the rate of momentum is equal to the impulse of an external force.

When the acceleration can be neglected then the Newton's law in its basic formulation $\sum \vec{F} = m\vec{a}$ simplifies to $\sum \vec{F} = 0$. This is the condition of static equilibrium. When the vector sum of all applied forces is equal to zero, then the body is said to be in a state of equilibrium. And that is the subject of statics in which bodies are stationary or move with respect to 'fixed stars'.

[4] The North-South bound rivers and the trade winds are good examples.

I4.2. Important terms to remember

Force might be understood as the cause of the change of motion.
Matter commonly exists in four states (or phases): *solid, liquid, gas,* and *plasma.* Matter has many properties as *volume, density, color, temperature,* and also *the mass* and the *weight.*

> **Mass** is the measure of unwillingness of the matter (body) to change its state of motion. It is independent of the gravitational field.
> **Weight** – another property of matter – depends, however, on the existence and intensity of gravitational field.

I4.3. SI metric units

The international systems of units SI (Le Système International d'unites) defines seven basic quantities. They are measured by units for which standard symbols (labels) are used. For more details see https://www.bipm.org/utils/common/pdf/si_brochure_8_en.pdf .

I4.3.1 Seven basic SI units are

Quantity	Unit	Symbol
length	meter	m
mass	kilogram	kg
time	second	s
electric current	ampere	A
thermodynamic temperature	kelvin	K
amount of substance	mole	mol
luminous intensity	candela	cd

I4.3.2 SI derived units used in mechanics

Derived quantity	Name	Symbol	In base units
area	square meter		m^2
volume	cubic meter		m^3
speed, velocity	meter per second		m/s
acceleration	meter per second squared		m/s^2
mass density	kilogram per cubic meter		kg/m^3
plane angle	radian	rad	1
frequency	hertz	Hz	s^{-1}
force	newton	N	$kg\,m\,s^{-2}$
pressure, stress	pascal	$Pa = N/m^2$	$kg\,m^{-1}\,s^{-2}$
energy, work	joule	$J = N\,m$	$kg\,m^2\,s^{-2}$
power	watt	$W = J/s$	$kg\,m^2\,s^{-3}$

It should be reminded that in literature, and even more frequently in real life, we can still encounter units of so-called **technical system of units** in which the force quantity was considered as the base unit while the mass quantity was a derived one. In this system the force is measured in units of [kp] – kiloponds and the mass, the derived unit, is measured in $[\mathrm{kp\,s^2/m}]$. This unit – in contradistinction to that defined in imperial units – has no name.
It is worth noticing that a sort of technical system, using, however, imperial units i.e. pound, feet, degree of Fahrenheit etc, is still in use the United States. The force is measured in pound-force [lbf] while the mass in pound-mass [lbm] units, called slug. For more details see www.en.wikipedia.org/wiki/Imperial_units

I4.4. Work, energy, power and corresponding units

I4.4.1. Mechanical work

In mechanics, the term work is used for something produced by physical effort. Mechanical work (work for short) is a scalar quantity defined as a dot product of two vectors, i.e. the force and the displacement. When both quantities are of variable nature we have to work with increments.

The increment of work is $\mathrm{d}W = \mathbf{F}^{\mathrm{T}}\mathrm{d}\mathbf{s} = \mathrm{d}\mathbf{s}^{\mathrm{T}}\mathbf{F} = \vec{F}\cdot\mathrm{d}\vec{s} = \left|\vec{F}\right|\left|d\vec{s}\right|\cos\varphi$,

where φ is the angle between vectors $\vec{a}$ and $\vec{b}$. If both components are constant and have the same line of action, then one can simply state that mechanical *work* = *force* $\times$ *displacement*.

I4.4.2. Mechanical energy

The mechanical energy (energy for short) is an ability to produce work. Energy and work are measured by the same units, i.e. joules [J]. The law of conservation of energy states that the total energy of an isolated system is conserved over time. Energy can be transformed from one form to another.

Units of work and energy in the SI system and their relation to the old technical system

$\mathrm{J} = \mathrm{Nm}$, joule = newton $\times$ meter	kpm,	kp $\times$ meter
$1\,\mathrm{J} = 0{,}102\,\mathrm{kpm}$	$1\,\mathrm{kpm} = 9{,}81\,\mathrm{J}$	
Recall, how it is related to the heat energy	$1\,\mathrm{kpm} = 2{,}343\,\mathrm{cal}$,	$1\,\mathrm{kcal} = 427\,\mathrm{kpm}$

I4.4.3. Mechanical power

Mechanical power (power for short) is the rate of work, or work exerted per unit of time, i.e. **power = work/time**. It is measured in watts $[\mathrm{W}]$.

a) Metric horsepower. See Fig. I04.

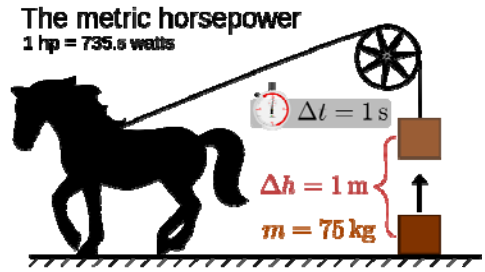


Fig. I04. Horsepower definition

$$\mathrm{W} = \mathrm{J/s}$$
$$1\,\mathrm{kW} = 1{,}36\,\mathrm{hp_{metric}}$$
$$\mathrm{Ws} = \mathrm{J}$$
$$1\,\mathrm{kWh} = 3{,}6\times10^{6}\,\mathrm{J} = 367\,000\ \mathrm{kpm}$$

So,

$1\,\mathrm{hp_{metric}} = 75\,\mathrm{kpm/s}$ ... metric horsepower,

$1\,\mathrm{hp_{metric}} = 0{,}736\,\mathrm{kW}$.

b) British horsepower

James Watt determined that a horse could turn a mill wheel 144 times in an hour; that is 2.4 times a minute. The wheel was 12 feet (3.6576 meters) in radius; therefore, the horse traveled 2.4·2π·12 feet in one minute. He judged that the horse could pull with a force of 180 force pounds. So

$$P = \frac{W}{t} = \frac{Fd}{t} = \frac{180\mathrm{lbf}\times2{,}4\times2\pi\times12\mathrm{ft}}{1\,\mathrm{min}} = 32{,}572\,\frac{\mathrm{lbf\ ft}}{\mathrm{min}}.$$

James Watt defined and evaluated the horsepower as 32,572 ft lbf/min, which was then rounded to 33,000 ft·lbf/min. The equivalent in SI units gives

$$1\,\mathrm{hp_{British}} = 33\ 000\ \mathrm{lbf\ ft/min} = 550\ \mathrm{lbf\ ft/s} \approx 17\ 696\ \mathrm{lbm\,ft^2\,s^{-3}} = 745{,}69987158227\ \mathrm{W}.$$

It slightly differs from the metric horse power. Take care when you buy a new car out of continental Europe.

I4.4.4. Potential and kinetic energy

If a particle of mass m, in the Earth's gravitational field, is raised to the height of h, then its potential energy E_p is defined as the work done W. So,

$W = E_\mathrm{p} = mgh$, where g is the gravitational acceleration.

We say that a particle, being raised to the height of h gathers the potential energy E_p.

If the particle is released (with zero initial velocity) from that elevated position, defined by h, it hits the initial position (ground) by velocity v, which might be determined from the equation of motion describing the free fall, using a few simple kinematic rules. We can write

$$ma = mg\,, \qquad \frac{\mathrm{d}v^2}{2\mathrm{d}x} = g\,, \qquad \int_0^v \mathrm{d}v^2 = 2g\int_0^h dx\,, \quad v^2 = 2gh \Rightarrow h = \frac{v^2}{2g}\,.$$

This way, we have obtained the relation between the 'hit' velocity and the height from which the particle was released.

The work 'obtained' by the falling particle from the height h is also mgh.

Substituting $h = \dfrac{v^2}{2g}$ into the previous equation we get the kinetic energy in the form

$$E_\mathrm{k} = mgh = \frac{1}{2}mv^2\,.$$

Neglecting the resistance, the sum of potential and kinetic energies, at any moment, is constant. For the rate of kinetic energy (for a mass particle), we can write

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \sum \mathbf{F}_i\,, \quad m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\mathrm{d}\mathbf{r} = \sum \mathbf{F}_i\,\mathrm{d}\mathbf{r}\,, \quad \text{but } \mathrm{d}\mathbf{r} = \mathbf{v}\,\mathrm{d}t\,, \qquad \text{so, } m\,\mathbf{v}\mathrm{d}\mathbf{v} = \sum \mathbf{F}_i\,\mathrm{d}\mathbf{r}\,,$$

$$m\int_{\mathbf{v}_0}^{\mathbf{v}} \mathbf{v}\mathrm{d}\mathbf{v} = \int \sum \mathbf{F}_i\,\mathrm{d}\mathbf{r} \quad \text{and finally } \frac{1}{2}m\left(\mathbf{v}^2 - \mathrm{v}_0^2\right) = W\,.$$

I4.4.5. A few things to remember

$E_\mathrm{k} - E_{\mathrm{k}0} = W$.
The change of kinetic energy (between the initial and final positions) is equal to the work done by applied forces.

Since the $work = power \times time$, then $\mathrm{d}W = P\,\mathrm{d}t$. Differentiating we get

$$dE_\mathrm{k} = P\,\mathrm{d}t \Rightarrow \frac{dE_\mathrm{k}}{\mathrm{d}t} = P\,.$$

The rate of kinetic energy is equal to the power of applied forces.

Also

$$work = force \times dispacement\,,$$

$$\frac{\mathrm{d}(work)}{\mathrm{d}t} = force \times \frac{\mathrm{d}(displacement)}{\mathrm{d}t}\,,$$

$$power = force \times velocity\,.$$

I4.5. Graphical engineering shorthand

The picture is worth a thousand words. That's why simple sketches are frequently used in the text to improve proper understanding of presented topics. Only a few samples with short explanations are presented in Fig. I05. The rest will be dutifully and systematically shown and explained later.

2D representation of axiradial and radial bearings.

2D rotary joint (constraint) connected to frame.

2D rotary-sliding joint connected to frame.

2D statically determinate truss bridge.

2D clamped beam.

Left – two rods (bars) connected by a rotary joint. Only axial forces could be transmitted.

Right – two welded beams. Axial forces, as well as bending moments, could be transmitted.

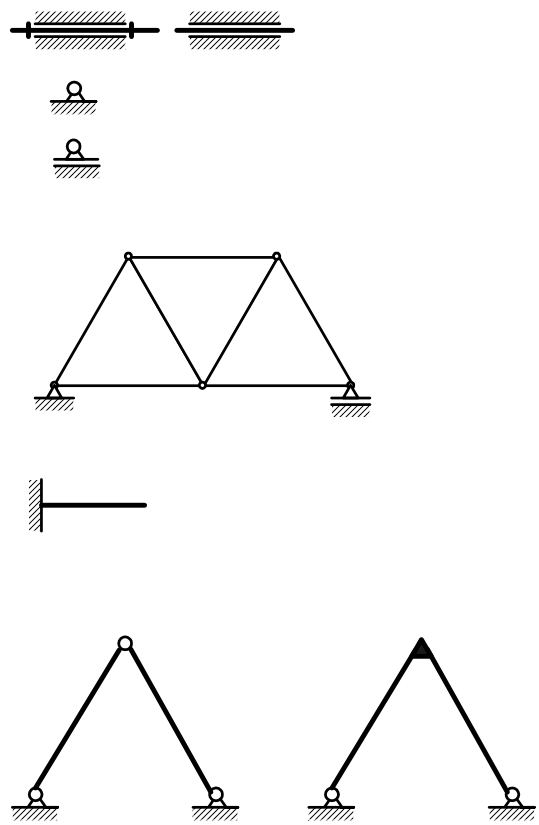


Fig. I05. Engineering shorthand

The schemes we are using are stripped to bare necessities as it is shown in following two pictures. The level of simplification varies according to actual purposes.

On the left, see Fig. I06, there is schematically depicted a crankshaft mechanism as it suits the needs for static analysis. Both crank and rod are simply represented by straight lines. The trajectories of the rod and piston pins are indicated. On the right, see Fig. I07 there is a slightly more complex representation of a four-stroke engine, of which the crankshaft mechanism is a crucial part. Still, it is a substantial simplification of an actual appearance of engine parts seen in Fig. I08.

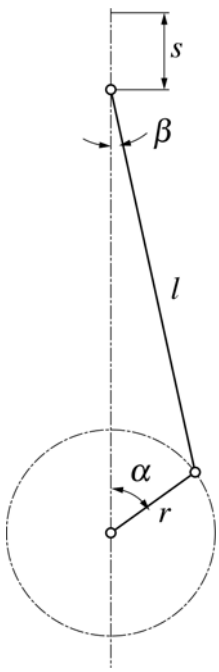


Fig. I06. Scheme of crankshaft mechanism

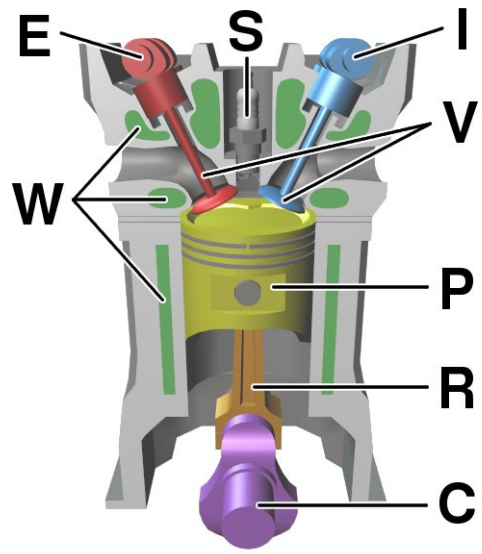


Fig. I07. Four-stroke engine
E – exhaust cam, S – spark
I – intake cam, W – water
P – piston, R – connecting rod
C – crank

Fig. I08. Connecting rod and piston – actual machine parts

Statics

Scope

1. Introduction to statics
2. Forces, moments, torque
3. Principle of transmissibility
4. Equilibrium
5. Equivalence
6. Degrees of freedom
7. Constraints and free body diagram
8. Classification of constraints
9. Friction
10. Rolling resistance
11. Principle of virtual work
12. Internal forces
13. Centre of gravity, centre of mass, and static moment
14. References

S1. Introduction to statics

In this text, the subject of statics is understood as a part of mechanics of rigid bodies. Statics deals with the analysis of static loads (forces and moments that do not vary in time) acting on rigid bodies trying to ascertain the conditions under which the equilibrium might occur. When in equilibrium, the bodies are either at rest or move with constant velocities. The condition of zero or constant velocity, i.e. $\vec{v} = 0$ or $\vec{v} = const$, actually means that the acceleration, the time derivative of velocity, is equal to zero, thus $\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = 0$. So, in static analysis, the time and acceleration play no role[1].

From it follows that Newton's law, in its simplest form, $\vec{F} = m\vec{a}$ written for a particle, degenerates to $\vec{F} = \vec{0}$. The last equation represents the condition of equilibrium requiring that the resulting force, or more generally the sum of all acting forces, should be identically equal to zero. For the equilibrium of bodies, the condition of zero moments has to be added. This will be explained later.

The reader is recommended to study other textbooks and web sources cited in Paragraph 14 of this chapter. Studying the texts of references listed there allows to broaden the reader's view on mechanics of rigid bodies. Following many worked-out examples might not only help to deepen understanding the subject of statics but also to increase the reader's proficiency needed to solve more complicated engineering tasks – to find out what is crucial and what might be neglected.

[1] Of course, all the phenomena occur in time. So, the subject of statics is a good approximation of those problems where bodies move so slowly, that their acceleration can be neglected.

S2. Forces, moments, torque

Definitions of quantities appearing in mechanics, as force, moment, pressure, stress, energy, etc, are rather intuitive and often circular. A few examples from standard textbooks are following.

Force *is only a name for the product of acceleration by mass.* Attributed to d'Alembert and cited in [1, p.532].

Forces *are vector quantities which are best described by intuitive concepts such as push or pull*. See [2].

Similar unsatisfying definitions may be found for time. Intuitively, everybody knows what it is until the moment when a direct and precise definition is required. See [3].

S2.1. Force

There is no precise definition of force. The force is usually defined by its effects. In the presented text we accept a simple, easily understood and intuitive definition, namely that the force represents an action of one body on another. This action is either due to an actual contact between bodies (the forces between interacting bodies are equal and opposite) or due to an action at a distance (for example due to the gravitational or the magnetic fields).

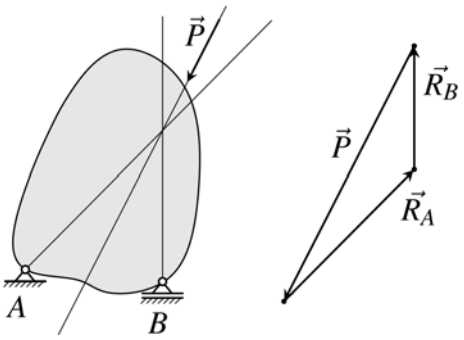


Fig. S01. Transmissible force

In most cases, the action between bodies is simplified as a point contact, even if actual contacts always occur in finite-size areas instead, and the actual 'action' is actually provided by pressure. So, we assume that forces are vector quantities represented by their directions and magnitudes as an applied force $\vec{P}$ shown in Fig. S01 with indicated reaction forces from the frame. We will explain that these forces are in equilibrium.

S2.2. Moment and couple

Generally, the moment of a force is a torque action of that force with respect to a point, or to an axis.

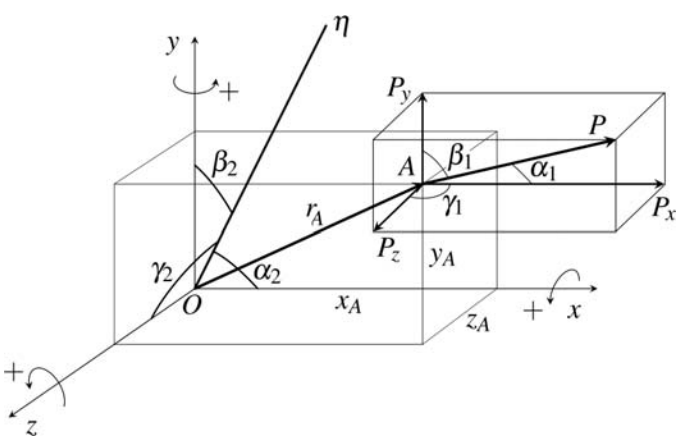


Fig. S02. Moment of a force

S2.3. Moment of a force about a point and about an axis

Moment of the force $\vec{P}$ about a point O, see Fig. S02, in the right-handed Cartesian coordinate system O, x, y, z is a vector, defined by means of the cross product

$$\vec{M}_{\mathrm{O}} = \vec{r}_{\mathrm{A}} \times \vec{P}, \qquad \text{(S2\_1)}$$

where $\vec{r}_{\mathrm{A}} = x_{\mathrm{A}}\vec{i} + y_{\mathrm{A}}\vec{j} + z_{\mathrm{A}}\vec{k}$ is the radius vector of the point of the application of the force $\vec{P}$, defined by $\vec{P} = P_x\vec{i} + P_y\vec{j} + P_z\vec{k}$. Its components are $P_x = \left|\vec{P}\right|\cos\alpha_1$, $P_y = \left|\vec{P}\right|\cos\beta_1$, $P_z = \left|\vec{P}\right|\cos\gamma_1$ and the magnitude of that force is $P = \left|\vec{P}\right| = \sqrt{P_x^2 + P_y^2 + P_z^2}$.

The cross product, defining the moment, is usually evaluated as a determinant by the Sarus'rule, i.e.

$$\vec{M}_{\mathrm{O}} = \vec{r}_{\mathrm{A}} \times \vec{P} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_{\mathrm{A}} & y_{\mathrm{A}} & z_{\mathrm{A}} \\ P_x & P_y & P_z \end{vmatrix} = \vec{i}\left(y_{\mathrm{A}}P_z - z_{\mathrm{A}}P_y\right) + \vec{j}\left(z_{\mathrm{A}}P_x - x_{\mathrm{A}}P_z\right) + \vec{k}\left(x_{\mathrm{A}}P_y - y_{\mathrm{A}}P_x\right) = \quad \ldots \text{(S2\_2)}$$

$$= M_x\vec{i} + M_y\vec{j} + M_z\vec{k}; \quad \text{Magnitude: } M_O = \left|\vec{M}_O\right| = \sqrt{M_x^2 + M_y^2 + M_z^2}.$$

The vector components of the moment are scalars and have geometrical meanings of moment components of that force about particular axes, i.e.

$$\begin{aligned} M_x &= \left(y_{\mathrm{A}}P_z - z_{\mathrm{A}}P_y\right), \\ M_y &= \left(z_{\mathrm{A}}P_x - x_{\mathrm{A}}P_z\right), \\ M_z &= \left(x_{\mathrm{A}}P_y - y_{\mathrm{A}}P_x\right). \end{aligned} \qquad \text{(S2\_3)}$$

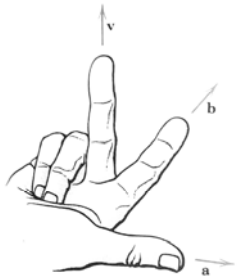


Fig. S03. Right-hand rule

The resulting vector is perpendicular to the plane formed by both components of the cross product and its positive direction is defined by the right-handed rule. The picture in Fig. S03 is for a triple of vectors $\vec{v} = \vec{a} \times \vec{b}$.

The positive sense of rotation of a moment about an axis, indicated by curved arrows (see Fig. S02), corresponds to a rotary motion of an imaginary nut, which causes its lateral motion along a right-handed thread, located along that axis, in the direction of the positive sense of that axis. Observing Fig. S04 we may also say that if $\vec{a}$ is rotated into the direction of $\vec{b}$ through an angle (less than π), then $\vec{v}$ advances in the same direction as a right-handed nut would if it turned in the same way.

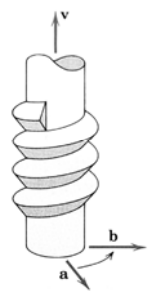


Fig. S04. Right-hand screw

The scalar value of the moment of $\vec{P}$ about a line η, defined by a unit vector $\vec{e}_\eta = \vec{i}\cos\alpha_2 + \vec{j}\cos\beta_2 + \vec{k}\cos\gamma_2$, is actually the projection of $\vec{M}_\mathrm{O}$ into that line. The projection is defined by the dot product multiplication, which gives

$$M_\eta = \vec{M}_\mathrm{O}\cdot\vec{e}_\eta = \left(M_x\vec{i} + M_y\vec{j} + M_z\vec{k}\right)\cdot\left(\vec{i}\cos\alpha_2 + \vec{j}\cos\beta_2 + \vec{k}\cos\gamma_2\right) = \\ = M_x\cos\alpha_2 + M_y\cos\beta_2 + M_z\cos\gamma_2 \,. \qquad \ldots \text{(S2\_4)}$$

Using the matrix notation, we can alternatively proceed as follows.

Defining the force $\mathbf{P} = \begin{Bmatrix} P_x \\ P_y \\ P_z \end{Bmatrix}$ as a column vector and the radius coordinate matrix by

$$\hat{\mathbf{r}} = \begin{bmatrix} 0 & -z_\mathrm{A} & y_\mathrm{A} \\ z_\mathrm{A} & 0 & -x_\mathrm{A} \\ -y_\mathrm{A} & x_\mathrm{A} & 0 \end{bmatrix},$$

then the matrix representation of the moment is a product of the radius coordinate matrix multiplied by the column vector of force components

$$\mathbf{M}_\mathrm{O} = \begin{Bmatrix} M_x \\ M_y \\ M_z \end{Bmatrix} = \hat{\mathbf{r}}\mathbf{P} = \begin{bmatrix} 0 & -z_\mathrm{A} & y_\mathrm{A} \\ z_\mathrm{A} & 0 & -x_\mathrm{A} \\ -y_\mathrm{A} & x_\mathrm{A} & 0 \end{bmatrix}\begin{Bmatrix} P_x \\ P_y \\ P_z \end{Bmatrix} = \begin{Bmatrix} y_\mathrm{A}P_z - z_\mathrm{A}P_y \\ z_\mathrm{A}P_x - x_\mathrm{A}P_z \\ x_\mathrm{A}P_y - y_\mathrm{A}P_x \end{Bmatrix}.$$

Sometimes, one can simply evaluate components of a moment by mere inspection. As an example, the acting force and its components are shown using the Monge's projection in Fig. S05.

Observing Fig. S05 we might immediately express the components of force moments about the indicated coordinate axes by inspection

$$M_x = -F_y(b - r\sin\psi),$$
$$M_y = F_x(b - r\sin\psi),$$
$$M_z = F_y(a + r\cos\psi) - F_x h\,.$$

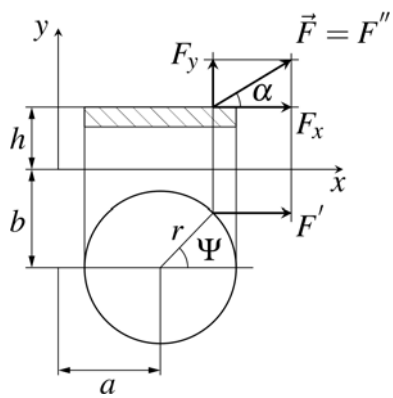


Fig. S05. Moment of a force

S2.4. Couple of forces

By a couple of forces (briefly just a couple) we understand two forces, say $\vec{F}$ and $-\vec{F}$, equal in magnitude and oppositely directed, acting on parallel lines that do not coincide. See Fig. S06. The resultant moment of that couple is a vector perpendicular to the plane formed by those parallel lines and its magnitude is $M_C = \left|\vec{M}_C\right| = Fr$, where r is the shortest distance between the parallel lines.

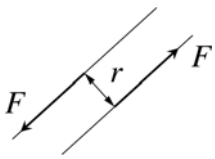


Fig. S06. Couple of forces

The moment of a couple is a free vector – in mechanics of rigid bodies, it can be located anywhere, while in mechanics of deformable bodies its location is crucial. The moment of a couple is often called a torque.

Earlier, for rigid bodies, we have stated that a force, as a bound vector attached to the line of action, can freely move along that line. However, it cannot, without penalty, be shifted laterally.

If one still has to shift the force laterally, then that action has to be compensated for by adding a couple. The rule is that a single force, acting along a specified line of action of a rigid body, can be replaced by an equal and parallel force F provided that a couple of forces is added in such a way that the moment of that couple is $M = Fd$, where d is the shortest distance between two lines of action.

Hint – what to do if we intend to shift a force laterally, say to the right

We add two parallel forces at the required position that are equal in magnitude and oppositely directed. In the rigid body world, nothing is changed since the forces are canceling themselves and are thus causing no overall effect.

Decomposing the middle part of the sketch, as indicated in Fig. S07, we might deduce that to shift a force laterally requires adding a proper couple, which – in this case – is oriented counterclockwise.

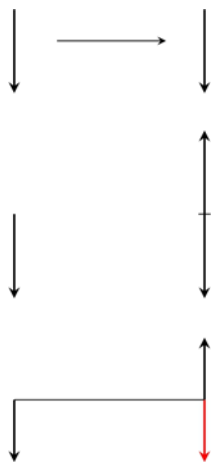


Fig. S07. Shift a force laterally

S3. Principle of transmissibility – is valid for rigid bodies only

The exact location of a force along its 'line of action' is immaterial. In our example, depicted in Fig. S08, the location of force $\vec{P}$ does not influence so-called reaction forces[2] acting on supports[3]. This is due to the fact that we assume that the bodies are perfectly rigid, i.e. not deformed due to applied forces. This principle does not apply to deformable bodies.

[2] How to evaluate reaction forces will be presented later.
[3] If a force were applied to a body which is not supported, the body would start to accelerate. This is, however, the problem that is out of the scope of statics – it belongs to the realm of dynamics.

If a body, shown in Fig. S08, is considered deformable, then the forces $\vec{P}_1$ and $\vec{P}_2$ cannot be taken as identical and their effects on the body are generally different. The subject will be treated later.

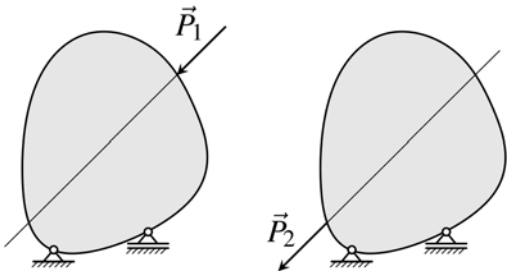


Fig. S08. In mechanics of deformable bodies the force is non-transmissible

S4. Equilibrium

A spatial system of forces and moments is in equilibrium if the sum of all forces and the sum of all moments are equal to zero. Then, we say that such a system is in the state of equilibrium. In vector form, we write

$$\sum \vec{F}_i = \vec{0}, \quad \sum \vec{M}_i = \vec{0}. \qquad \text{(S4\_1)}$$

S5. Equivalence

Any system of forces can be replaced by an equivalent force, called the resultant force, such as $\vec{R} = \sum \vec{F}_i$. (S5_1)

As an alternative, the force can also be replaced by an equivalent system consisting of a single force at a chosen point, say O, and of a corresponding moment, as illustrated in Fig. S09.

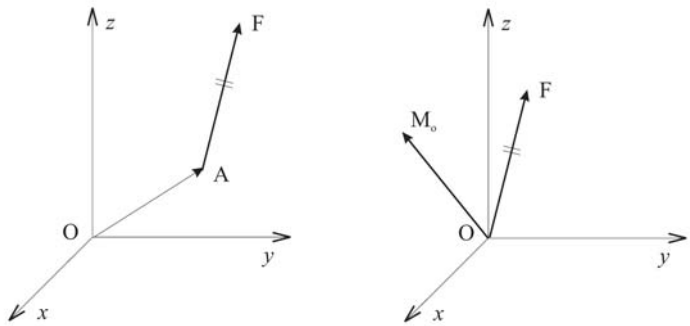


Fig. 09. Force-couple equivalence

So, any force system can be replaced either by a single equivalent force or by a force at a chosen location accompanied by a properly dimensioned couple.

For practical purposes, it is convenient to treat equilibrium and equivalence conditions for 1D, 2D and for 3D cases separately.

The simplest situation occurs when there are no moments and **all the forces share a single line of action**.

Two forces $\vec{Z}$ and $\vec{P}$, shown in Fig. S10, are in equilibrium if $\vec{Z} + \vec{P} = 0$. The condition of equilibrium – expressed in a scalar form – is: $-Z_x + P_x = 0$. In this case, the index, denoting the axis, is arbitrary, immaterial and might be omitted.

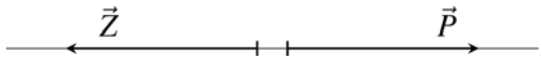


Fig. S10. Equilibrium of two forces

Forces pass through a single point in 2D space

Equivalence

Two forces $\vec{Z}_1, \vec{Z}_2$, shown in Fig. S11, are acting at the single point in a plane. The force $\vec{V}$ is the resultant force. It is equivalent to forces $\vec{Z}_1, \vec{Z}_2$. The force $\vec{P}$ is in equilibrium with the force $\vec{V}$. The condition of equivalence, written in vector and scalar notations, is

$$\vec{V} = \vec{Z}_1 + \vec{Z}_2,$$
$$V_x = Z_{1x} + Z_{2x}, \quad V_y = Z_{1y} + Z_{2y}.$$

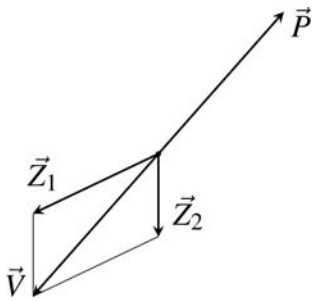


Fig. S11. Equilibrium and equivalence

Equilibrium

The force $\vec{P}$, see Fig. S11 again, being of the same size and of the opposite direction with respect to the force $\vec{V}$, is said to be in equilibrium with force $\vec{V}$ or with its components $\vec{Z}_1, \vec{Z}_2$. The condition of equilibrium, written sequentially in vector and scalar notations, is

$$\vec{P} + \vec{V} = 0,$$
$$P_x + V_x = 0, \quad P_y + V_y = 0.$$

The difference between equivalence and equilibrium, as treated graphically, is depicted in Fig. S12.

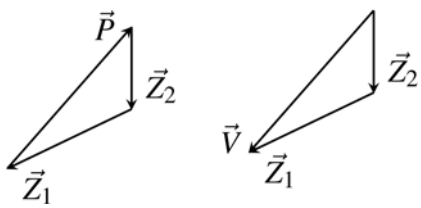


Fig. S12. Equilibrium – left, equivalence – right

Summary of equilibrium conditions for forces and moments, i.e. $\sum \vec{F}_i = \vec{0}, \quad \sum \vec{M}_i = \vec{0}$, expressed in scalar forms for different spatial cases

System of forces acting along a single line of action

$$\sum F_i = 0. \tag{S4_2}$$

System of forces acting at a single point in plane

$$\sum F_{xi} = 0, \quad \sum F_{yi} = 0. \tag{S4_3}$$

For a *system of forces and moments in a plane* to be in equilibrium, two component-type equations (sum of all the forces along the specified directions is to be zero) and one moment type equation (sum of all moments of all forces about a specified point is to be zero) has to be satisfied.

$$
\begin{aligned}
x: &\quad \sum F_{xi} = 0, \\
y: &\quad \sum F_{yi} = 0, \\
M_{\mathrm{A}}: &\quad \sum F_{xi} y_i + F_{yi} x_i + M_i = 0.
\end{aligned}
\tag{S4_4}
$$

Out of three equilibrium conditions, at least one equation of the moment type always has to be used. Using three component-type equations leads to a linearly dependent system of equations that is singular and does not allow finding a unique solution. Each component type equation could, however, be replaced by a moment one. But not vice versa.

System of forces for a single point in 3D

$$
\sum F_{xi} = 0, \quad \sum F_{yi} = 0, \quad \sum F_{zi} = 0 . \tag{S4_5}
$$

System of forces and moments for a body in 3D

$$
\begin{aligned}
x: &\quad \sum F_{xi} = 0, \\
y: &\quad \sum F_{yi} = 0, \\
z: &\quad \sum F_{zi} = 0, \\
M_{\mathrm{x}}: &\quad \sum F_{yi} z_i + F_{zi} y_i + M_{xi} = 0, \\
M_{\mathrm{y}}: &\quad \sum F_{zi} x_i + F_{xi} z_i + M_{yi} = 0, \\
M_{\mathrm{z}}: &\quad \sum F_{xi} y_i + F_{yi} x_i + M_{zi} = 0.
\end{aligned}
\tag{S4_6}
$$

Out of six equilibrium conditions, at least three equations of the moment type have to be always used.

S6. Degrees of freedom

The number of degrees of freedom (number of dof's for short) is the measure of a degree of 'movability'[4] of a body. The number of degrees of freedom of a rigid body is defined as the number of independent coordinates uniquely determining the position of that body in space.

A few examples might clarify the subject.

- The position of a free[5] **rigid body in space** is uniquely determined by six coordinates – three longitudinal coordinates of a certain point (usually the center of mass) and three rotational coordinates (angles) determining the body orientation (pitch, yaw and roll angles) with respect to arbitrarily chosen fixed coordinate axes. We say that a free rigid body in space has **six dof's**.

- The position of a free **rigid body in a plane** is uniquely determined by three coordinates – two longitudinal coordinates of a certain point (usually the center of

[4] The term mobility is used as well.
[5] The attribute 'free' indicates that the body in question is unsupported. We might also say that a free body is not constrained. As for example a space capsule in the outer space.

mass) and one rotational coordinate (angle) determining the body orientation with respect to chosen coordinate axes. So, the free rigid body in a plane has **three dof's**.

- The position of a **particle**[6] **in space** is uniquely determined by three longitudinal coordinates – **three dof's.**

- The position of a **particle in plane** is uniquely determined by two longitudinal coordinates – **two dof's**.

- The position of a **particle constrained to a line** is determined by one positional coordinate – it has **one dof**.

The concept of degrees of freedom for deformable bodies is quite different and will be treated and explained later.

S7. Constraints and free body diagram

From a rather academic treatment of equilibrium of forces, we have analyzed so far, we proceed to the treatment of a body, or to a set of bodies, that are in a state of equilibrium. As before, the condition of equilibrium requires that the vector sums of all the forces and all the moments, acting on the body or bodies, are equal to zero. Strictly speaking, we are seeking the conditions under which the state of equilibrium might occur.

We have already mentioned that a free body is an object not being supported – it is freely 'flying' in space and has its degree of 'movability' which is specified by the number of degrees of freedom. A free body, however, cannot be treated by static tools because any applied nonzero force would invoke its motion with certain acceleration. Since the acceleration and time are excluded from considerations in statics, a body always has to be 'properly' constrained – i.e. connected to the frame or to other bodies.

By the mechanical constraint, we understand a type of a mechanical attachments gadget or implement, having a specific engineering design, allowing the bodies to be constrained (restricted) in their motions or allowing them a sort of limited motion. In most cases, we will be evaluating the constraint forces and moments due to applied forces for bodies staying in rest and having zero dof's.

A body can be constrained in its potential motions by a variety of ways. Among the analyzed bodies there is always one playing a special role, namely the fixed frame of reference (frame for short) which is firmly attached to the ground – usually to Mother Earth, which for most of analyzed cases is considered stationary.

In statics, the analyzed problem might consist of one body attached to the frame or of a set of interconnected bodies. Generally, whenever a motion of one body is restricted by another body, including the fixed frame, then there are corresponding forces and/or moments, typical for the type of constraint in question, occurring in contact (connecting) locations.

[6] A particle is rigid by definition. It has no dimensions and its angular orientation in space is immaterial.

To allow the mathematical analysis of applied forces that are in equilibrium with the constraint forces (often called reactions) a helpful tool, named the free-body diagram (FBD), is frequently used.

Free body diagram is a graphical sketch used to visualize body (bodies) under applied forces and moments and also under the reaction forces and reaction moments occurring due to the existence of particulate constraints. This helps to understand the way how the bodies are mutually connected facilitating thus the formulation of equilibrium equations.

The free body diagram depicts the forces and moments applied to a body, and complement them with corresponding reaction forces and moments. This is a sort of mental procedure. The actual physical connections (constraints) between bodies are apparently removed and replaced by equivalent forces and moments that are characteristic for the particular type of constraints in question. These forces and moments should be suitably indicated and named to be susceptible for further analysis. This way, we convert the problem of bodies being in the state of equilibrium to that of equilibrium of forces.

Example – a car on an inclined plane

Given: A stationary car on the inclined road, being held in its position by a rope, is schematically depicted in Fig. S13. The car brakes are not applied. The mass of the car is m.
Determine: Using the free body diagram technique, visualize the forces acting on the car, write the equilibrium equation and find the force in the rope, say S, required to hold the car in its current position.

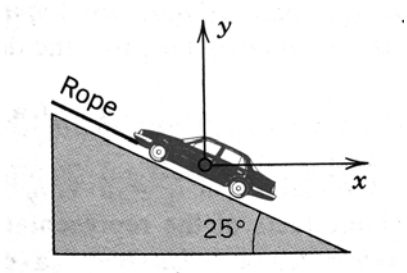


Fig. S13. Car on the slope

The thought process required for establishing the free-body diagram is illustrated by a sketch in Fig. S14. As the first approximation, the car might be considered as a particle through which all the forces pass. Then, the first constraint, the 'road', is removed and replaced by an equivalent reaction force acting from the road to the car.

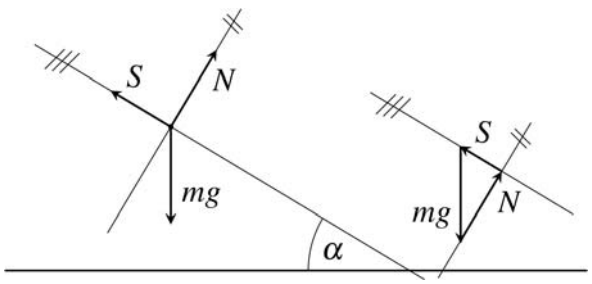


Fig. S14. Equilibrium of forces

Considering the stationary car and neglecting friction effects, the reaction force, say N, has to be perpendicular to the 'road'. That force allows the car to stay in its current position even if the road is 'removed'. The second constraint, the rope, is cut and replaced by a force, say S, acting in the direction of the rope. What remains to consider is the weight of the car, say $W = mg$, which can be visualized by a vertical vector, acting 'down', in the opposite direction of the y-axis.

We have added constraint forces (reactions), named them, and now the equilibrium conditions can be mathematically expressed. In this simple case, all three forces pass through a single point, approximating the car[7].

For the car remain stationary, all three forces have to be in equilibrium – their vector sum has to be zero, so

$$\vec{S} + \vec{N} + m\vec{g} = \vec{0}\,.$$

Since we have simplified the problem by assuming that all the forces pass through a single point, then the scalar conditions of equilibrium (equivalent to the above vector form) might be written for *x* and *y* force components in the form

$$x: \quad -S\cos\alpha + N\sin\alpha = 0,$$
$$y: \quad S\sin\alpha + N\cos\alpha - mg = 0.$$

This way, we have obtained two linear algebraic equations. Knowing the angle α and the weight of car *mg*, two unknowns, i.e. N (the normal reaction) and S (the force in the rope) can easily be determined.

The magnitudes and directions of unknown vectors $\vec{N}$ and $\vec{S}$ can also be determined graphically, as indicated on the right-hand side of Fig. S14. The graphical reasoning is also based on the fact that the resulting force of these three vectors is equal to zero – satisfying thus the conditions of equilibrium.

There are different kinds of constraints (body connections). To determine the character of forces and/or moments, associated with a particular type of constraint, is the subject of the following text.

S8. Classification of constraints

At first, frictionless constraints are considered. See the chapter devoted to friction phenomena.

S8.1. Rigid constraint – clamping

This kind of constraint is assumed to be perfectly rigid – it secures that in the connection point there is no motion possible between the fixed frame (the wall) and the body shown in Fig. S15.

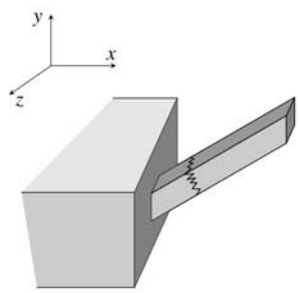


Fig. S15. Clamped beam

[7] Accepting this degree of simplification, there is no way how to determine the forces between the road and wheels. There are, however, other manners, by which we will solve this task.

Let's analyze 2D and 3D situations separately.

a) Rigid constraint in 2D – clamping

To draw a free body diagram for a clamped beam in a plane requires removing the rigid connection constraint, where the beam is attached to the frame by clamping. Simultaneously we require that the beam stays in its current position. To do so, we have to add a force (having two scalar components) preventing the beam to move in up and down and in sideways directions. Also, a moment has to be added preventing the beam to rotate. (The moment vector has one scalar component). A free body in a plane has three degrees of freedom. Each force component removes one possible motion – we say that two force components remove two translatory degrees of freedom. The remaining degree of freedom, i.e. the rotation, is removed by the reaction moment. As before, the equilibrium equations are $\sum \vec{F}_i = \vec{0}, \quad \sum \vec{M}_i = \vec{0}$, and their scalar form is

$$
\begin{aligned}
x: &\quad \sum F_{xi} = 0, \\
y: &\quad \sum F_{yi} = 0, \\
M_A: &\quad \sum F_{xi} y_i + F_{yi} x_i + M_i = 0,
\end{aligned}
$$

where M_i is the $i-$th applied moment.

Example – clamped beam in 2D

Given: In Fig. S16 there is schematically depicted a 2D beam of the length l which is clamped at point C to the rigid frame, being visualized by hatching. Graphically, the beam is approximated by a straight horizontal line of the length l. The beam is loaded by forces F_1 and F_2 at locations indicated by the distance dimensioning a and l. The forces are graphically represented by vectors with their directions and magnitudes defined. Also, a moment M_1 is applied at the location of force F_1. In the lower part of the figure there is shown the free body diagram corresponding to this case.

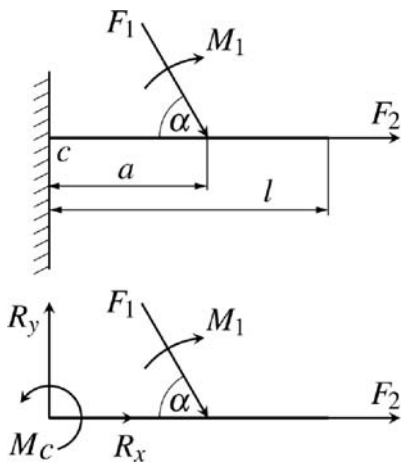


Fig. S16. Free body diagram for a loaded clamped beam

Besides the external loading, represented by F_1, F_2 and M_1, there is the reaction force, represented by its two components, and the reaction moment. These correspond to the rigid connection (clamping) between the beam and the frame. As explained before the reaction force and moment are associated with this type of constraint in question – the clamping. The reaction force components, say R_x, R_y, and the reaction moment, say M_C, are unknown quantities that are to be determined from equilibrium conditions:

$$
\begin{aligned}
&\text{sum of force components in } x \text{ direction}: && R_x + F_1 \cos\alpha + F_2 = 0, \\
&\text{sum of force components in } y \text{ direction}: && R_y - F_1 \sin\alpha = 0, \\
&\text{sum of force moments about the point C}: && M_C - F_1 a \sin\alpha - M_1 = 0.
\end{aligned}
$$

Determine: Knowing dimensions, angle α, external forces F_1, F_2 and external moment M_1 the above equations could be solved for the unknown reaction components, i.e. R_x, R_y and M_C.

b) Rigid constraint in 3D – clamping

A free body in 3D space has six degrees of freedom. In this case, the clamped constraint represents also a vector force and a vector moment, but these, however, represent three force components and three moment components – altogether six unknown reactions.

S8.2. Rotary constraint – hinge joint or pin joint or revolute pair

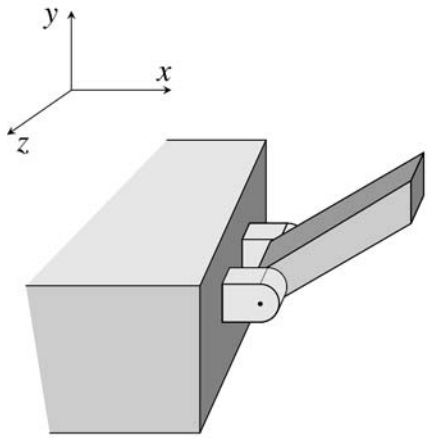


Fig. S17. A hinge constraint

A hinge constraint allows a free rotation only about the hinge axis and prevents any translation. See Fig. S17.

In 2D, a single reaction force (with two scalar components) represents this constraint. In 3D, a single reaction force (with three scalar components) and two reaction moment components, i.e. M_x, M_y, about axes perpendicular to the hinge axis, are needed. In 3D these five components remove five of dof's corresponding to a free body in space, and since $6-5=1$, there remains one degree of freedom corresponding to the rotation about the *z*-axis.

Graphical representation of different types of constraints in free body diagrams

In the text and in accompanying examples we will use a sort of easily drawn 'shorthand' representations of constraints. In Fig. S18 a few of them are shown together with reaction forces and moments that correspond to a particular type of constraint and are needed for the free body diagram reasoning.

An example of an engineering design of a shaft supported by two bearings is in Fig. S19.

The left bearing, being firmly connected to the shaft and to the housing, is able to support both radial and axial forces. The right bearing, connected to the shaft but allowing left or right sliding motions with respect to the housing, permits to support radial forces only.

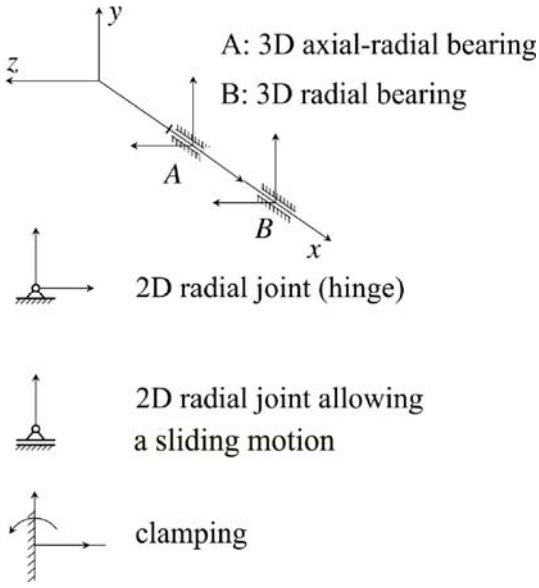


Fig. S18. Free body diagrams

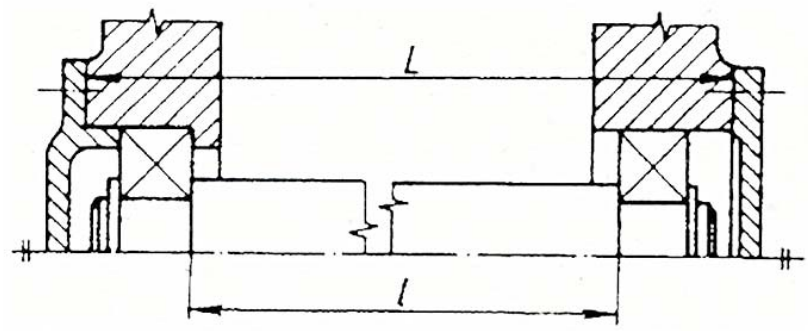


Fig. S19. Engineering design of a supported shaft

This is the way how the thermal expansion of the shaft is provided for.

The corresponding FBD is in Fig. S20.

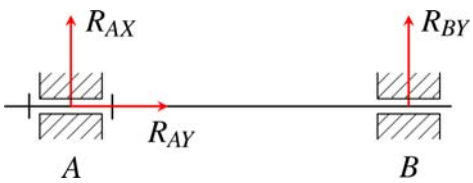
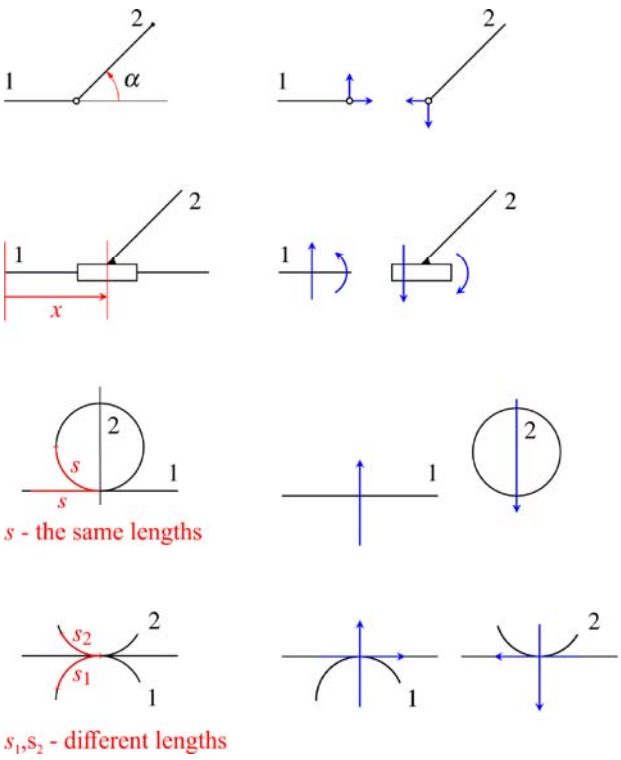


Fig. 20. FBD for a supported shaft

Individual bodies of mechanical structures are connected by constraints of different types, sometimes also called kinematic pairs. Generally, a kinematic pair is a connection between two bodies imposing constraints on their relative motions. A few types of 2D frictionless kinematic pairs are listed in Table 1.

Planar kinematic pairs



	Revolute pair, joint – allows rotary motion only, 1 dof, 2 reaction components.
	Prismatic pair, slider, sleeve – allows translational motion only, 1 dof, 2 reaction components.
	Rolling pair – no slipping, 1 dof, 1 reaction component.
	Higher pair – slipping occurs, 2 dofs, 2 reaction components

Table. 1. Kinematic pairs

Hint - kinematic pairs, the principle of action and reaction, FBD for a 2D crank mechanism

The crank mechanism has one dof. See Fig. S21. So, only one coordinate (either the angular displacement of the crank or the positional coordinate of the piston) is sufficient for determining its actual position.

Crank, denoted by number 2, is a 2D body loaded by a planar system of forces. Thus, three equilibrium equations are required.

Rod, number 3, even if it is actually a body in the plane, is loaded by forces sharing the same line of action. So, only one equilibrium condition is needed.

Piston, number 4, is a 2D body loaded by a planar system of forces. Three equilibrium equations are required.

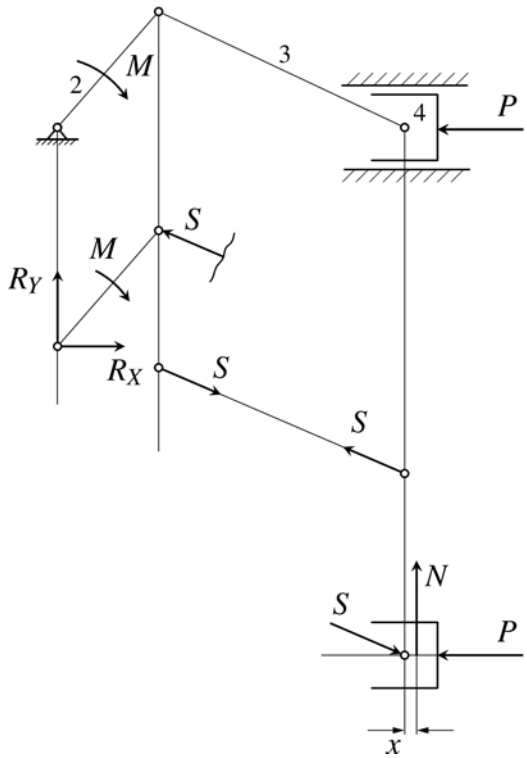


Fig. S21. Free body diagram for crank mechanism

Generally, the normal force between the cylinder and piston does not pass through the piston pin.

S8.3. Ball and socket joint

This type of constraint allows for attachment of two bodies, allowing their free mutual rotation and at the same time restricting any mutual translation. See Fig. S22. A human hip joint is a good example. When considered in a free body diagram, this type of constraint is replaced by a reaction force having three components in 3D and two in 2D. Since a free frictionless rotation is allowed, there are no moment components in this case.

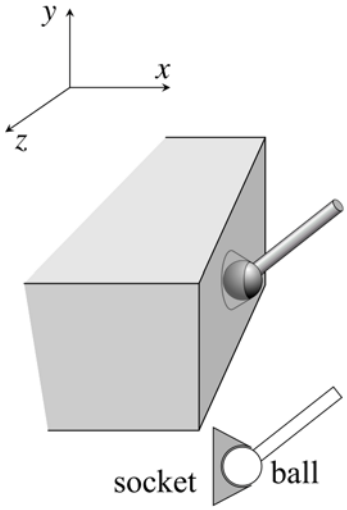


Fig. S22. A spherical joint or a socket ball

S8.4. Wires, ropes, cables, chains, rods, bars, struts, springs, belts, and dashpots

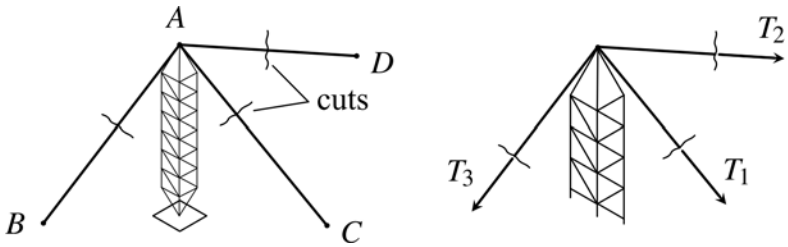


Fig. S23. Free body diagrams for a rope

Wires, ropes, cables, chains, rods, bars, struts, springs, and dashpots are machine design elements that are frequently used in mechanical engineering. See Fig. S23. They serve as connecting elements, whose transversal dimensions are small with respect to lateral ones. That's why their transversal dimensions, their weight and/or mass are often neglected. The element of this kind is only able to transfer the force that acts within the line connecting its extremity points. In rigid body mechanics, they are considered inextensible.

Wires, ropes, cables, chains, and belts are assumed to have a capability transmitting tension forces only, while rods, bars, struts, springs, and dashpots could transfer compression forces as well. The terms rod, bar and strut are considered synonymous.

S8.5. Springs

The spring is a machine design element that is elongated under the influence of an axial tensional force or shortened when an axial compression force is applied. See Fig. S24. Its initial or unstrained length is l, the change of length[8] (which might be positive or negative), due to an applied force might be denoted Δl. The force in the spring, say F, is usually taken as a linear function of elongation, i.e. $F = k\Delta l$. The coefficient of linearity, k, goes under the name of the stiffness, or the spring stiffness. Its dimension is $[\mathrm{N/m}]$. The spring linearity should not be taken for granted, it is valid only for cases when the elongation Δl is small with respect to the unstrained length l, and for cases when elastic deformations, with no permanent material changes (so the plasticity effects are excluded) in the spring, occur.

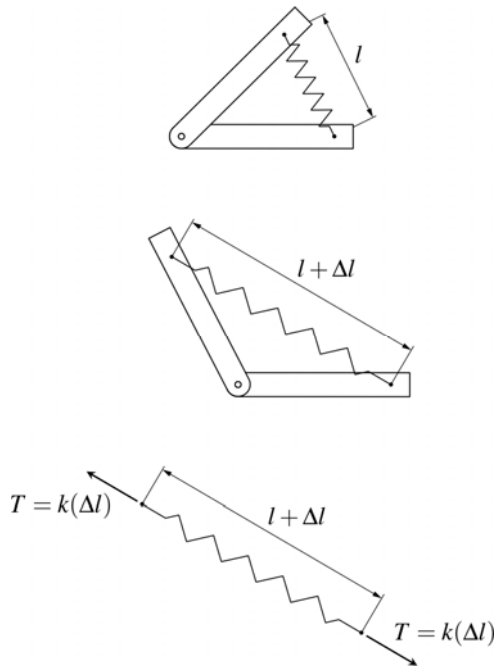


Fig. S24. Free body diagram for a spring

In engineering, we also encounter torsional springs and coiled springs. The latter is still used in mechanical watches being connected to the balance wheel securing thus its regular oscillations.

The actual appearance of a spiral spring is in Fig. S25.

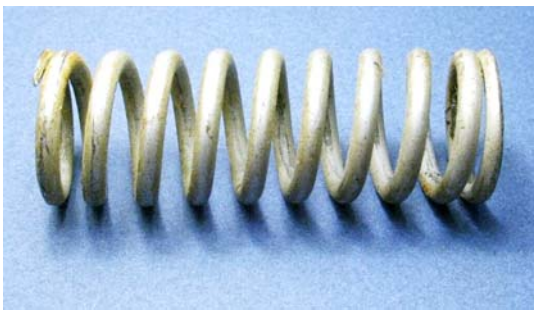


Fig. S25. Actual spring

[8] Sometimes called the deformation or elongation.

The meaning of spring linearity is graphically illustrated in Fig. S26. This kind of behavior is in accordance with Hooke's law, that states that the force F, required to elongate or shorten a spring by a displacement x, is linearly proportional to the magnitude of that displacement, i.e. $F = kx$, where the coefficient of proportionality k is called the spring stiffness.

The law is named after Robert Hooke who published it in 1676 in the form *ut tensio, sic vis,* meaning 'as the extension, so the force'.

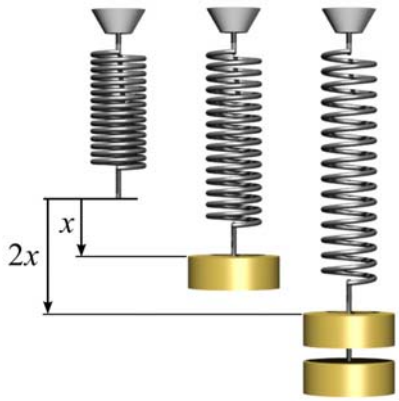


Fig. S26. Linear behavior of a spring

It should be emphasized that Hooke's law is only a first-order approximation of the real response of springs. Spring characteristics, i.e. the dependence of force to spring elongation, could be of various types as shown in Fig. S27. That is (1) progressive, (2) linear, (3) degressive, (4) almost constant or (5) progressive with a knee.

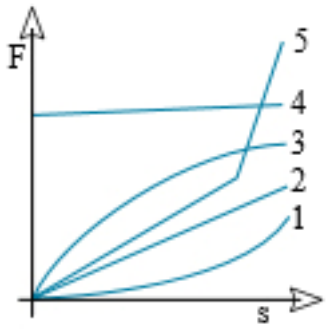


Fig. S27. Spring characteristics

S8.6. Dashpots, dampers

The dashpot, also called damper, is a machine design element that resists the change of its initial length l. The resisting (reaction) force is linearly proportional to the change of its initial length, or by other words, to the relative velocity v_R of its extremity points. See Fig. S28.

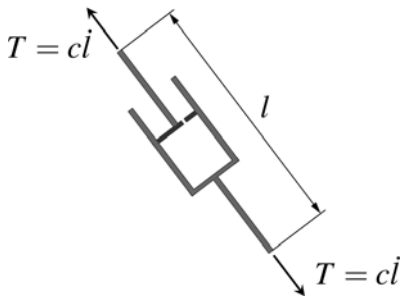


Fig. S28. Free body diagram for a dashpot

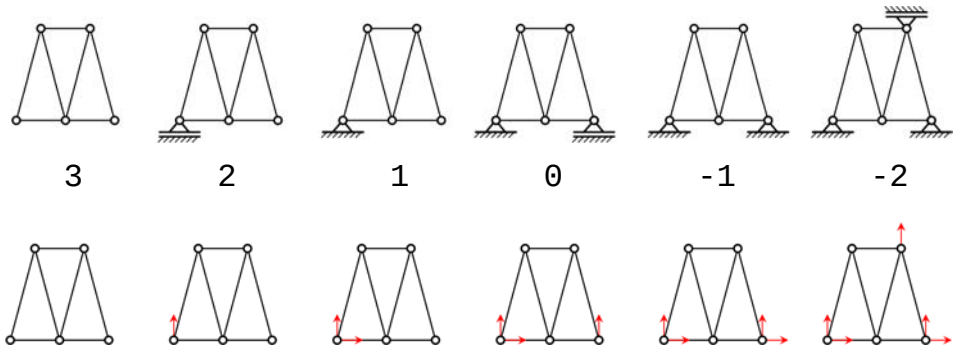
So, the corresponding reaction force appearing in the FBD is $T = c\dfrac{\mathrm{d}l}{\mathrm{d}t} = c\dot{l} = cv_R$.

In this case, a linear behavior of the dashpot is assumed. Often, non-linear dashpots, with forces proportional to the second, third and higher powers of velocity, are considered in engineering practice as well.

The dashpots play no role in statics. We will explain their importance in dynamics.

Survey for constraints, FBD and dof's

Six cases of a differently constrained body (a truss structure, composed of thin rods, also called bars) connected at their ends by frictionless joints, are depicted in Table 2. Due to miscellaneous constraints applied to that body, we can analyze six different cases with different numbers of degrees of freedom. For simplicity, the bridge structure is assumed to be two dimensional and all the constraints are considered frictionless.



Structure						
#dof's	3	2	1	0	-1	-2
reactions						
# reactions components	0	1	2	3	4	5
# equilibrium equations	3	3	3	3	3	3
type of structure	..... moving			properly constrained	... constrained ... too much	
type of problem	..... statically ..... underdeterminate			statically determinate	... statically ... interdeterminate	
to be solved in	..... dynamics			statics	strength of material	

Table 2. Degrees of freedom and free body diagrams

The first column corresponds to a free, unconstraint or unsupported body that has 3 dof's in the plane. There are no reaction forces to be associated with the case.

The second column. The body is attached to the frame by a radial joint that besides the rotation allows left or right sliding motions. By mutual consent, the vertical motion in the up direction is prohibited. The body could freely rotate around the joint and also could freely move in left or right directions as well, it thus has two dof's. In the FBD this joint could be replaced by one unknown reaction component on the left, which would act vertically.

The third column. The body is attached to the frame by a radial joint allowing a free rotation around this joint only, it thus has one dof. In the FBD, this joint could be replaced by two unknown components of the reaction force in that joint.

The constraint bodies, depicted in the first three columns, have one common property, – they can move. Generally, the moving structures are characterized by the fact that their number of dof's is greater than zero. Mechanical systems composed of more rigid elements, having a positive number of dof's, are often called mechanisms. More about the subject is in the chapter devoted to kinematics.

Any structure able to move will start to change its position in space and cannot be treated by statics tools of mechanics. Their motions, due to the applied forces and moments, are described not by equations of equilibrium, but by equations of motions having the form of ordinary differential equations. In the following text, we will show how these problems are analyzed by tools of dynamics.

The fourth column. The body is attached to the frame at two places. On the left, there is a radial joint, which when considered alone, allows a free rotation. On the right, there is a sliding radial joint allowing both the rotation and the horizontal motions. The left joint removes one dof, and represents two unknown reaction components, the right one two dof's and requires to add one unknown reaction component in the FBD. Altogether, the body cannot move and has, in this case, zero degrees of freedom. Reaction forces represent three unknowns, two on the left and one on the right, and for a body in a plane, we have three scalar equations of equilibrium at our disposal. This case is thus easily solvable. We say that such a system is *statically determinate*.

Generally, we can state that the actual number of dof's of a body, say i, plus the number of unknown reaction components due to prescribed constraints, say m, is equal to the number of dof's of that body "freely" flying in the space (rigid body motions). In plane, we could write $i + m = 3$, in space $i + m = 6$.

The fifth and sixth columns correspond to structures that from the statics point of view are 'constrained too much'. They have a negative number of degrees of freedom. We say that these cases are *statically interdetermine*. In these cases, the number of unknown reaction components is greater than the number of available equilibrium equations. Consequently, the conditions of equilibrium do not suffice to find unknown reactions. Cases of this kind will be explained, analyzed and treated in chapters devoted to the mechanics of deformable bodies. We will show that adding an adequate number of so-called deformation conditions, the tasks of this type can be solved.

The treated tasks could be classified according to the number of degrees of freedom.
If # dof's = 0, then the mechanical system is said to be *statically determinate* and for given forces and moments, the corresponding reactions are readily obtained from properly formulated equilibrium conditions. In this case, the system is stationary and the number of unknowns is equal to the number of available equilibrium conditions.

If # dof's > 0, then the system is *statically underderterminate* and generally cannot be solved by statics tools. For given forces and moments, the system would start to move with accelerations and could only be treated by dynamics tools. Still, the tasks of this kind could be analyzed in statics if the problem is reformulated.

There are two possibilities.

First, for a given position determine such forces and moments that allow the system to stay in its current configuration.

Second, determine such a configuration in which the system – for a sufficient number of prescribed loads – will be in the state of equilibrium.

If # dof's < 0, then the system is said *statically indeterminate* and cannot be solved by statics tools since the number of unknown reactions is greater than the number of available equilibrium equations. The tasks of this kind could be treated by tools of mechanics of deformable bodies, where a suitable number of so-called deformation conditions are added, which together with equilibrium equations will suffice to find all the unknown reactions.

Example – structure of six rods, zero dof's, forces passing through a point

Type of task: 2D, rods, forces passing through a point.

Given: dimensions, angles, force Q.
Determine: rod forces S_1 to S_6.

A structure, composed of six rod elements that are connected by frictionless joints, is depicted in Fig. S29.

The left side and right side joints connect the structure to the fixed frame, which is indicated by hatching.

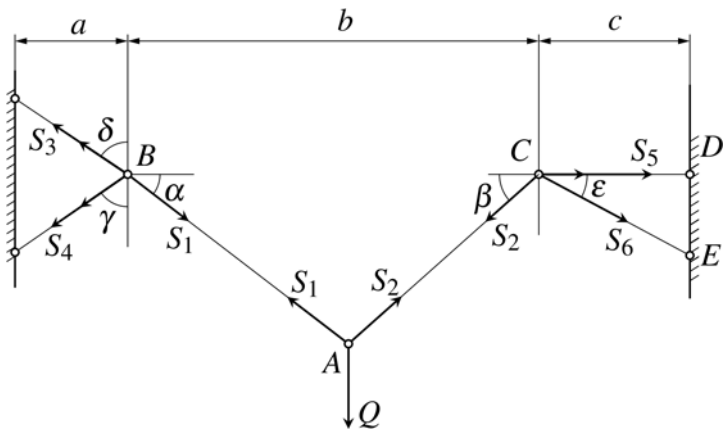


Fig. 29. Rod structure

The rods are able to transfer axial forces in directions of their end joints only, so to find them it is required to analyze the equilibrium of forces passing through the joints A, B and C, respectively. The vectors of all the forces have directions of rods (lines connecting their end joints), their directions, which might be chosen arbitrarily, are indicated by arrows.
Generally, the free body diagram, the principle of action and reaction and conditions of equilibrium are applied. In detail, we proceed in four steps. See Fig. S30.

a) Starting at the joint A we mentally cut the rods that are connected by a frictionless joint A and replace them by equivalent forces S_1 and S_2.

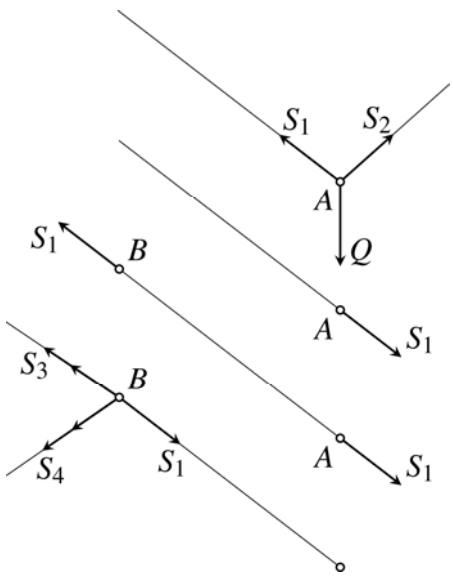


Fig. S30. Free body diagram, action and reaction, equilibrium

Their directions are given by lines connecting their end joints, their directions, indicated by arrows, are chosen arbitrarily. This is what we also see in the lower part of Fig. S29. Now, the conditions of equilibrium of forces acting at the joint A are applied – it is required that $\vec{S}_1 + \vec{S}_2 + \vec{Q} = 0$. Solving the equation allows determining the unknown forces S_1, S_2. So far, we are talking about forces acting on the joint A.

b) Now, let's analyze the forces acting on the rod with end joints A and B. According to the principle of action and reaction the joint A, as a part of the rod is acted on from the joint A itself by a force which has the same magnitude as before, but is of an opposite direction.

c) Equilibrium of forces acting on the rod AB. Since there are no other external forces acting on the rod, the left and right reaction forces have to have the same magnitude and the opposite directions to satisfy the equilibrium conditions.

d) Plotting the FBD for the joint B we take into account the principle of action and reaction again. Then the equilibrium conditions can be written. It is required that $\vec{S}_1 + \vec{S}_3 + \vec{S}_4 = 0$. For practical purposes, the vector equations of equilibrium are often replaced by a corresponding number of scalar equations.

Similarly, we proceed for other joints. We believe that a detailed discussion of this kind will not be needed when solving the tasks that follow.

Expressing the equilibrium conditions in scalar forms we can write

Forces passing through the joint A

$x: \quad -S_1 \cos\alpha + S_2 \cos\beta = 0,$

$y: \quad S_1 \sin\alpha + S_2 \sin\beta - Q = 0.$

– – – – – – – – – – – – – – – – – –

Knowing $Q \Rightarrow S_1, S_2$.

Forces passing through the joint B

$x: \quad S_1 \cos\alpha - S_3 \sin\delta - S_4 \sin\gamma = 0,$

$y: \quad -S_1 \sin\alpha + S_3 \cos\delta - S_4 \cos\gamma = 0.$

– –

Knowing $S_1 \Rightarrow S_3, S_4$.

Forces passing through the joint C

$x: \quad -S_2 \cos\beta + S_5 + S_6 \cos\varepsilon = 0,$

$y: \quad -S_2 \sin\beta - S_6 \sin\varepsilon = 0.$

Knowing $S_2 \Rightarrow S_5, S_6$.

We have stated that the directions of forces in FBD are chosen arbitrarily. Their actual directions come from analyzing the results of the numerical solution. If the resulting unknown variable has a positive value, then the original choice of direction was chosen correctly. And vice versa.

Example – forces passing through a single point. See Fig. S31.

Type of task: 3D, rods, zero dof's, forces passing through a point.

Given: dimensions, force *P*.
Determine: forces S_1 to S_3.

Three rods are attached to a rigid wall (plane *xz*) by frictionless joints. Their other ends are connected in another joint where a vertical force *P* is applied.

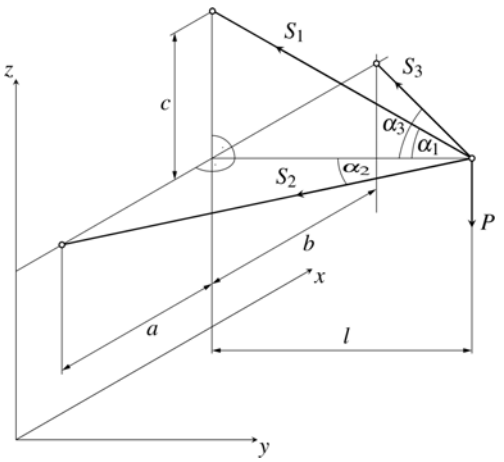


Fig. S31. Forces through a point

The angles come from geometry considerations, i.e. from
$\tan\alpha_1 = c/l, \quad \tan\alpha_2 = a/l, \quad \tan\alpha_3 = b/l$.

The task requires solving the spatial system of forces passing through a single point. As explained above three scalar equations in directions of coordinate axes are needed.

$$x: \quad -S_2\sin\alpha_2 + S_3\sin\alpha_3 = 0,$$
$$y: \quad -S_1\cos\alpha_1 - S_2\cos\alpha_2 - S_3\cos\alpha_3 = 0,$$
$$z: \quad S_1\sin\alpha_2 - P = 0.$$

– –

Knowing $P \Rightarrow S_1, S_2, S_3$.

Equilibrium conditions in the matrix notation are

$$\begin{bmatrix} 0 & -\sin\alpha_2 & \sin\alpha_3 \\ -\cos\alpha_1 & -\cos\alpha_2 & -\cos\alpha_3 \\ \sin\alpha_1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ P \end{Bmatrix}.$$

To get a purely analytical solution the Matlab symbolic toolbox might help. See the program `S01_three_rods_3D.m`

```
%S01_three_rods_3D
% old file is named tri_pruty_3D

clear
syms a1 a2 a3 P b A x
A = [0 -sin(a2) sin(a3); -cos(a1) -cos(a2) -cos(a3); sin(a1) 0 0];
b = [0; 0; P];
inv_A = inv(A);
x = A\b;
pretty(x)
```

The result is

```
[                      P
[                   -------                          ]
[                   sin(a1)                          ]
[                                                    ]
[               sin(a3) cos(a1) P                    ]
[- ----------------------------------------------    ]
[  sin(a1) (sin(a2) cos(a3) + sin(a3) cos(a2))       ]
[                                                    ]
[               sin(a2) cos(a1) P                    ]
[- ----------------------------------------------    ]
[  sin(a1) (sin(a2) cos(a3) + sin(a3) cos(a2))       ]
```

Example – forces passing through a single point of a body

Type of task: 2D body, zero dof's, all the forces are passing through a single point.

Given: dimensions, angles, forces *F*, *Q*.
Determine: reactions N_A, N_B.

A cylinder, whose weight is *Q*, is supported by two perpendicular planes, as depicted in Fig. S32, and loaded by a force *F*. Free body diagram reasoning requires to remove supporting planes and to add corresponding reactions, say N_A, N_B, acting at contact points. Since no friction is considered both reactions are perpendicular to supporting planes.

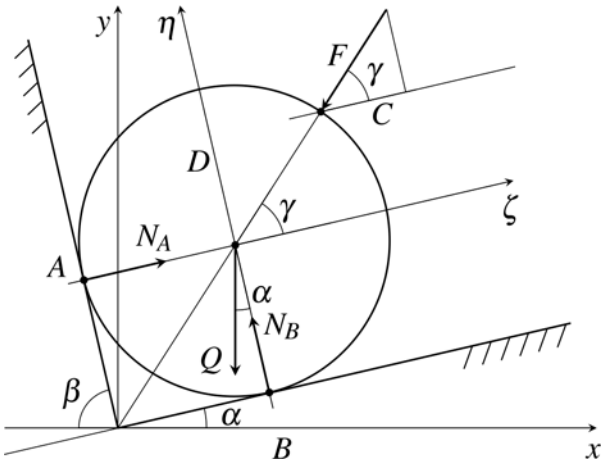


Fig. S32. Equilibrium of forces passing through a point

At the first sight, we deal with a body loaded by a planar system of forces, requiring expressing and solving three equilibrium equations. In this case, however, all the forces pass through a single point, so only two equilibrium conditions are needed.

Generally, the orientation of the coordinate system is arbitrary but a smart choice is always advantageous.

Equilibrium conditions, written for scalar components of acting forces, are

$$\xi: \quad N_A - Q\sin\alpha - F\cos\gamma = 0,$$
$$\eta: \quad N_B - Q\cos\alpha - F\sin\gamma = 0.$$

– – – – – – – – – – – – – – – – –

$$\Rightarrow N_A, N_B.$$

Discussion

The task would have no solution if the force F had an opposite direction. Mathematically, this would be indicated by negative values of contact reactions. If the angle $\gamma \neq \pi/4$ then the above solution is not valid. Explain why.

Example – forces acting on a 2D block, zero dof's

A rectangular block is supported by a sliding joint at point A (one vertical reaction) and by two rods connecting the block to the frame. See Fig. S33. Both rods have frictionless joints at their ends. Rod reactions represent axial forces in direction of their end joints. The block is loaded by forces P, Z and Q.

Type of task: 2D, body, zero dof's.

Given: P, Q, Z, dimensions, angles.
Determine: S_1, S_2, R_A.

Equilibrium equations

$$x: \quad -S_1 - S_2 \sin\beta - P\cos\alpha = 0,$$
$$y: \quad -S_2 \cos\beta - P\sin\alpha + Z - Q + R_A = 0,$$
$$M_B: -S_2 h \sin\beta - 2Pl\sin\alpha + Zb + R_A(2l - a) - Ql = 0.$$

Knowing $P, Q, Z, \alpha, \beta \Rightarrow S_1, S_2, R_A$.

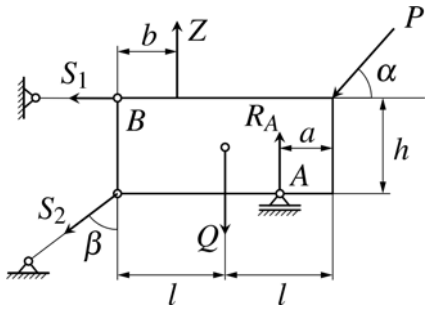


Fig. S33. Equilibrium of a body

Discussion

For certain combination of values and directions of forces P and Q the reaction R_A might be negative. What does it indicate?

Example – simplified 2D bridge

Type of task: A 2D structure with zero dof's, composed of three bodies is depicted in Fig. S34.

Given: dimensions and Q_2, Q_3, Q_4, α.
Determine: reactions at joints R_A, R_B, R_C, R_D, R_E.

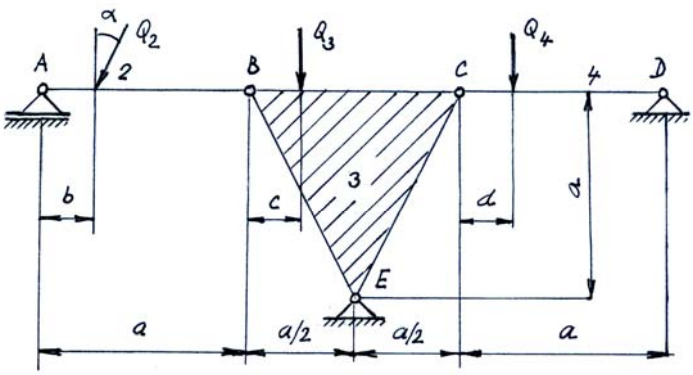


Fig. S34. A simplified bridge

Equilibrium of each body is treated separately. Notice the 'transfer' of reactions from one body to another using the principle of action and reaction as shown in Fig. S35.

Body 2

$$\text{Eq. 1:}\quad x:\quad -Q_2\sin\alpha + R_2 = 0,$$
$$\text{Eq. 2:}\quad y:\quad R_1 - Q_2\cos\alpha + R_3 = 0,$$
$$\text{Eq. 3:}\quad M_A:\quad -Q_2 b\cos\alpha + R_3 a = 0.$$

Body 3

$$\text{Eq. 4:}\quad x:\quad -R_2 + R_4 + R_6 = 0,$$
$$\text{Eq. 5:}\quad y:\quad -R_3 - Q_3 + R_5 + R_7 = 0,$$
$$\text{Eq, 6:}\quad M_B:\quad -Q_3 c + R_5 a + R_6 a + R_7\frac{a}{2} = 0.$$

Body 4

$$\text{Eq. 7:}\quad x:\quad -R_4 + R_8 = 0,$$
$$\text{Eq. 8:}\quad y:\quad -R_5 - Q_4 + R_9 = 0,$$
$$\text{Eq. 9:}\quad M_C:\quad -Q_4 d + R_9 a = 0.$$

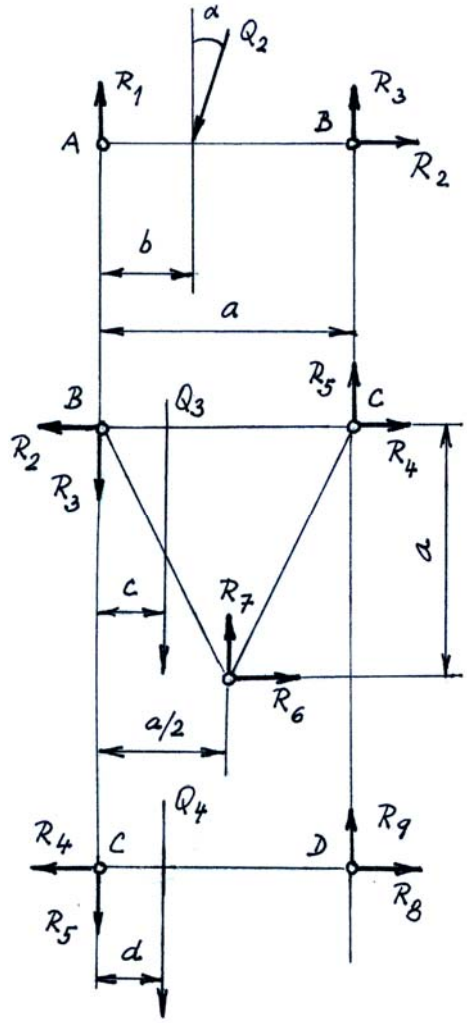


Fig. S35. Free body diagrams for bodies.

Now, follow the text of the program `S02_bridge.m`. Altogether, we have nine equations allowing to evaluate nine unknown reactions R_1 to R_9. The above equations could be written in the matrix form as $[\mathrm{KK}]\{\mathrm{R}\} = \{\mathrm{F}\}$. The matrix of the system of equilibrium equations is

```
%      1  2  3  4   5   6  7   8  9
KK  = [0  1  0  0   0   0  0   0  0;  %1
       1  0  1  0   0   0  0   0  0;  %2
       0  0  a  0   0   0  0   0  0;  %3
       0 -1  0  1   0   1  0   0  0;  %4
       0  0 -1  0   1   0  1   0  0;  %5
       0  0  0  0   a   a  a/2 0  0;  %6
       0  0  0 -1   0   0  0   1  0;  %7
       0  0  0  0  -1   0  0   0  1;  %8
       0  0  0  0   0   0  0   0  a]; %9
```

Right hand side – the vector of loading forces

```
F = [Q2*sin(alfa) Q2*cos(alfa) Q2*b*cos(alfa) 0 Q3 Q3*c 0 Q4 Q4*d]';
```

Reactions are obtained solving the system of algebraic equations

```
R = KK\F;
```

Results

```
reactions
            1       259.81
            2          500
            3       606.22
            4       853.11
            5         -800
            6      -353.11
            7       2906.2
            8       853.11
            9         1200
```

```
% S02_bridge
% old file name is mst_010_most_c2
clear; format short g
a = 1; b = 0.7; c = 0.2; d = 0.6;
Q2 = 1000; Q3 = 1500; Q4 = 2000;
alfa = pi/6;
% loading forces
F = [Q2*sin(alfa) Q2*cos(alfa) Q2*b*cos(alfa) 0 Q3 Q3*c 0 Q4 Q4*d]';
% the matrix and the right hand side of the system {KK}{R} = {Q}
%      1   2  3  4  5   6   7  8  9
KK  = [0  1  0  0   0   0  0   0  0;  % 1    Q2*sin(alfa)
       1  0  1  0   0   0  0   0  0;  % 2    Q2*cos(alfa)
       0  0  a  0   0   0  0   0  0;  % 3    Q2*b*cos(alfa)
       0 -1  0  1   0   1  0   0  0;  % 4    0
       0  0 -1  0   1   0  1   0  0;  % 5    Q3
       0  0  0  0   a   0  a/2 1  0;  % 6    Q3*c
       0  0  0 -1   0   0  0   1  0;  % 7    0
       0  0  0  0  -1   0  0   0  1;  % 8    Q4
       0  0  0  0   0   0  0   0  a]; % 9    Q4*d
rank(KK);                                    % check
R = KK\F;                                    % solution
counter = [1:9]';
disp('reactions')
disp([counter R])
```

Example – parallel forces in 3D

A block of weight Q is suspended by three parallel rods (connected to the block and to the frame by frictionless joints) of equal length as depicted by means of Monge's projection in Fig. S36.

Type of task: 3D, body.

Given: Q, dimensions.
Determine: rod forces S_1, S_2, S_3.

Here, we are dealing with a system of forces in 3D space, so 6 equilibrium scalar conditions are required.

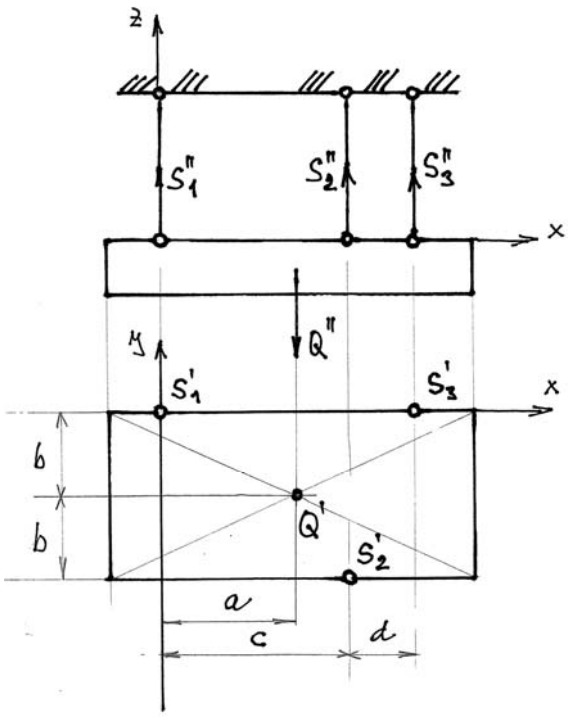


Fig. S36. Equilibrium of a body

$$
\begin{aligned}
&x: \quad 0 = 0,\\
&y: \quad 0 = 0,\\
&z: \quad -S_1 - S_2 - S_3 + Q = 0,\\
&M_x: Qb - 2S_2 b = 0,\\
&M_y: Qa - S_2 c - S_3(c+d) = 0,\\
&M_z: 0 = 0.
\end{aligned}
$$

Due to the fact that all the forces are parallel and vertical, three equations are satisfied identically. Knowing Q and dimensions, the remaining three equations suffice to evaluate unknown forces S_1, S_2, S_3.

Explain, why the task could not be solved if the block were suspended by more than three rods.

What would happen if the force Q were not vertical? Answer: The block would start to move and the task would not be solvable by statics tools.

Example – cable forces in 3D

Type of task: 3D, rods, zero dof's, forces passing through a point.
Given: Three rods, attached by frictionless joints to the 'ceiling', as depicted in Fig. S38, are connected by another joint located at point A. The system is loaded by an attached cylinder whose weight is *mg*.
Determine: rod forces.

It should be reminded that the direction cosines of a vector see Fig. S37, can be expressed in the form $\cos\varphi_x = a_x / \|\vec{a}\|$, $\cos\varphi_y = a_y / \|\vec{a}\|$, $\cos\varphi_z = a_z / \|\vec{a}\|$, where $\|\vec{a}\| = \sqrt{a_x^2 + a_y^2 + a_z^2}$.

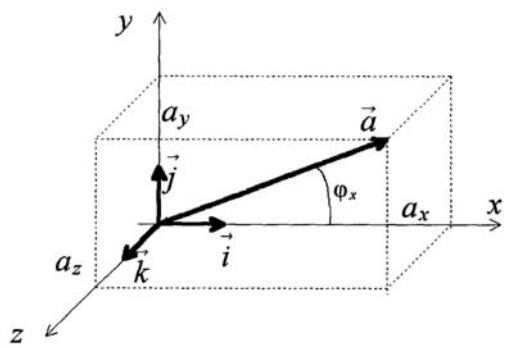


Fig. S37. Components of a vector.

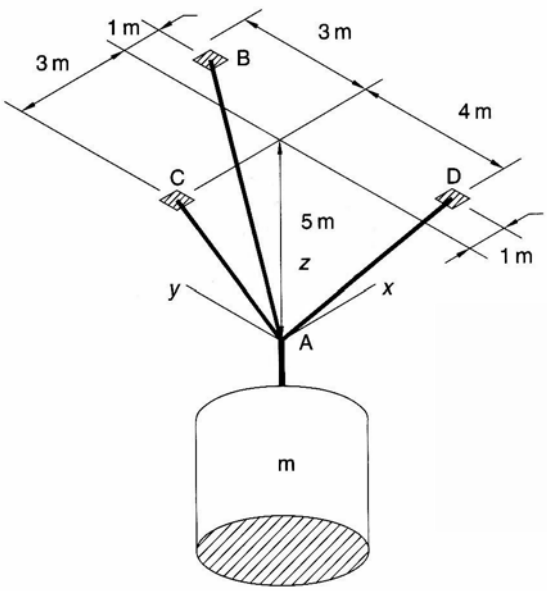


Fig. S38. Equilibrium of a body in space

Locating the origin of the coordinate system, as indicated in Fig. S38, then the radius vectors associated with rods, expressed in Matlab style, are

```
AB = [1 3 5];
AC = [-3 0 5];
AD = [1 -4 5];

% their lengths
L_AB = sqrt(dot(AB,AB));
L_AC = sqrt(dot(AC,AC));
L_AD = sqrt(dot(AD,AD));

% direction cosines for rod AB
cos_alfa(1) = AB(1)/L_AB;
cos_alfa(2) = AB(2)/L_AB;
cos_alfa(3) = AB(3)/L_AB;

% direction cosines for rod AC
cos_beta(1) = AC(1)/L_AC;
cos_beta(2) = AC(2)/L_AC;
cos_beta(3) = AC(3)/L_AC;

% direction cosines for rod AD
cos_gama(1) = AD(1)/L_AD;
cos_gama(2) = AD(2)/L_AD;
cos_gama(3) = AD(3)/L_AD;
```

Assembling them into a matrix of direction cosines columnwise

```
CS = [cos_alfa' cos_beta' cos_gama']
```

we get

```
CS =

    0.1690   -0.5145    0.1543
    0.5071         0   -0.6172
    0.8452    0.8575    0.7715
```

The equilibrium conditions are

$$
\begin{aligned}
x:&\quad T_1\cos\alpha_1 + T_2\cos\beta_1 + T_3\cos\gamma_1 = 0,\\
y:&\quad T_1\cos\alpha_2 + T_2\cos\beta_2 + T_3\cos\gamma_2 = 0,\\
z:&\quad T_1\cos\alpha_3 + T_2\cos\beta_3 + T_3\cos\gamma_3 - mg = 0.
\end{aligned}
$$

To simplify the subsequent analysis of results we have substituted $mg = 1$ here.

$$
\begin{bmatrix} \cos\alpha_1 & \cos\beta_1 & \cos\gamma_1 \\ \cos\alpha_2 & \cos\beta_2 & \cos\gamma_2 \\ \cos\alpha_3 & \cos\beta_3 & \cos\gamma_3 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \{b\}, \text{ where } \{b\} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}.
$$

Solving the system of equations by `T = CS\b` we get

```
T =
    0.5071
    0.2915
    0.4166
```

Now, we claim that the resulting rod forces are multiples of the value of mg .

Alternatively, we can proceed more efficiently, even if in a less transparent way. Let's collect the radius vectors in a matrix columnwise as

```
r(:,1) = [1 3 5];
r(:,2) = [-3 0 5];
r(:,3) = [1 -4 5];
```

Their lengths are

```
for i = 1:3
 LL(i) = sqrt(dot(r(:,i),r(:,i)));
end
```

Similarly, the direction cosines are stored columnwise into another matrix as

```
for i = 1:3
   CSS(:,i) = r(:,i)/LL(i);
end
```

The rest of the procedure is the same as before. This could be verified by executing the statement `TT = CS\b`. See the program `S03_cable_forces`.m.

```
% S03_ cable_forces
% m_024_cable_forces_en.m
clear
% position vectors
AB = [1 3 5];
AC = [-3 0 5];
AD = [1 -4 5];

% their lengths - Pythagoras and the dot product
L_AB = sqrt(dot(AB,AB));
L_AC = sqrt(dot(AC,AC));
L_AD = sqrt(dot(AD,AD));

% direction cosines for AB
cos_alfa(1) = AB(1)/L_AB;
cos_alfa(2) = AB(2)/L_AB;
cos_alfa(3) = AB(3)/L_AB;
% direction cosines for AC
cos_beta(1) = AC(1)/L_AC;
cos_beta(2) = AC(2)/L_AC;
cos_beta(3) = AC(3)/L_AC;
% direction cosines for AD
cos_gama(1) = AD(1)/L_AD;
cos_gama(2) = AD(2)/L_AD;
cos_gama(3) = AD(3)/L_AD;

CS = [cos_alfa' cos_beta' cos_gama']
% losding vector
b = [0 0 1]';
% solve the system oquations
T = CS\b

% and now, a more efficient style
% direction vectors are assembled columnwise in r matrix
r(:,1) = [1 3 5]; r(:,2) = [-3 0 5]; r(:,3) = [1 -4 5];

for i = 1:3                      % compute lengths
 LL(i) = sqrt(dot(r(:,i),r(:,i)));
end

% the same for direction cosines
for i = 1:3
   CSS(:,i) = r(:,i)/LL(i);
end

TT = CS\b                        % compute forces

% end of m_024_cable_forces.en
```

Exa mple – 2D body of weight W suspended by two cables, as seen in Fig. S39.

Type of task: forces passing through a point.
Given: $L_{\mathrm{BC}} = 5\ \mathrm{m}$, $L_{\mathrm{AB}} = 3\ \mathrm{m}$,
$D = 6\ \mathrm{m}$, $W = 2000\ \mathrm{N}$.
Determine: forces in cables.

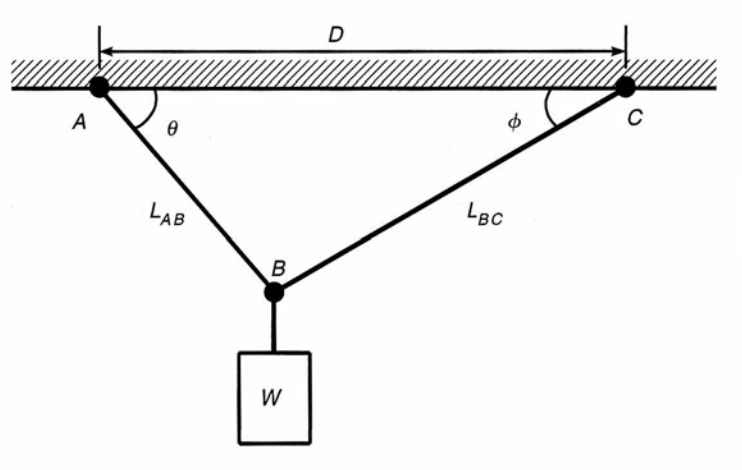


Fig. S39. Weight suspended by two cables

The equilibrium conditions for forces passing through the point B are

$$x: \quad -T_{\mathrm{AB}}\cos\theta + T_{\mathrm{BC}}\cos\phi = 0,$$
$$y: \quad +T_{\mathrm{AB}}\sin\theta + T_{\mathrm{BC}}\sin\phi - mg = 0.$$

Sine and cosine rules give

$$L_{\mathrm{BC}}\sin\theta = L_{\mathrm{AB}}\sin\phi,$$
$$L_{\mathrm{BC}}^2 = D^2 + L_{\mathrm{AB}}^2 - 2DL_{\mathrm{AB}}\cos\theta$$

and from this we get

$$\sin\phi = \frac{L_{\mathrm{AB}}\sin\theta}{L_{\mathrm{BC}}},$$
$$\cos\theta = \frac{D^2 + L_{\mathrm{AB}}^2 - L_{\mathrm{BC}}^2}{2DL_{\mathrm{AB}}}.$$

System of equations corresponding to equilibrium conditions is

$$\begin{bmatrix} -\cos\theta & \cos\phi \\ \sin\theta & \sin\phi \end{bmatrix} \begin{Bmatrix} T_{\mathrm{AB}} \\ T_{\mathrm{BC}} \end{Bmatrix} = \begin{Bmatrix} 0 \\ mg \end{Bmatrix}$$

Input data are

```
D = 6; LAB = 3; LBC = 5; MG = 2000;
```

Writing and executing this piece of code

```
theta = acos((D^2 + LAB^2 - LBC^2)/(2*D*LAB));
fi = asin(LAB*sin(theta)/LBC);

K = [-cos(theta) cos(fi); sin(theta) sin(fi)];
F = [0 MG]';
T = K\F
```

we obtain

```
T =
       1737.2
       1113.6
```

Now, let's analyze what would happen if, leaving the length of the rope L_{BC} constant, the angle ϕ is allowed to vary. Consequently, the length of L_{AB} will be varying as well. Now, in the enlarged task one has to determine the rope forces as functions of the varying length L_{AB}.

Considering the triangle properties and the condition that ropes cannot transmit compression forces we could write

$$L_{\mathrm{AB\_min}} = D - L_{\mathrm{BC}},$$
$$L_{\mathrm{AB\_min}} = \sqrt{D^2 + L_{\mathrm{BC}}^2}.$$

In Matlab we have

```
LAM_min = D - LBC;
LAB_max = sqrt(D^2 + LBC^2);
incr = 0.1;
LAB_range = LAM_min+incr : incr : LAB_max; % vynech zacatek intervalu
i = 0;
for LAB = LAB_range
    i = i + 1;
    theta = acos((D^2 + LAB^2 - LBC^2)/(2*D*LAB));
    fi = asin(LAB*sin(theta)/LBC);
    K = [-cos(theta) cos(fi); sin(theta) sin(fi)];
    F = [0 MG]';
    T = K\F;
    T_all(i,:) = T;
end
```

The program `S04_ weight_supported_by_two_cables` generates the plot showing the rope forces as functions of the varying length L_{AB}. See Fig. S40.

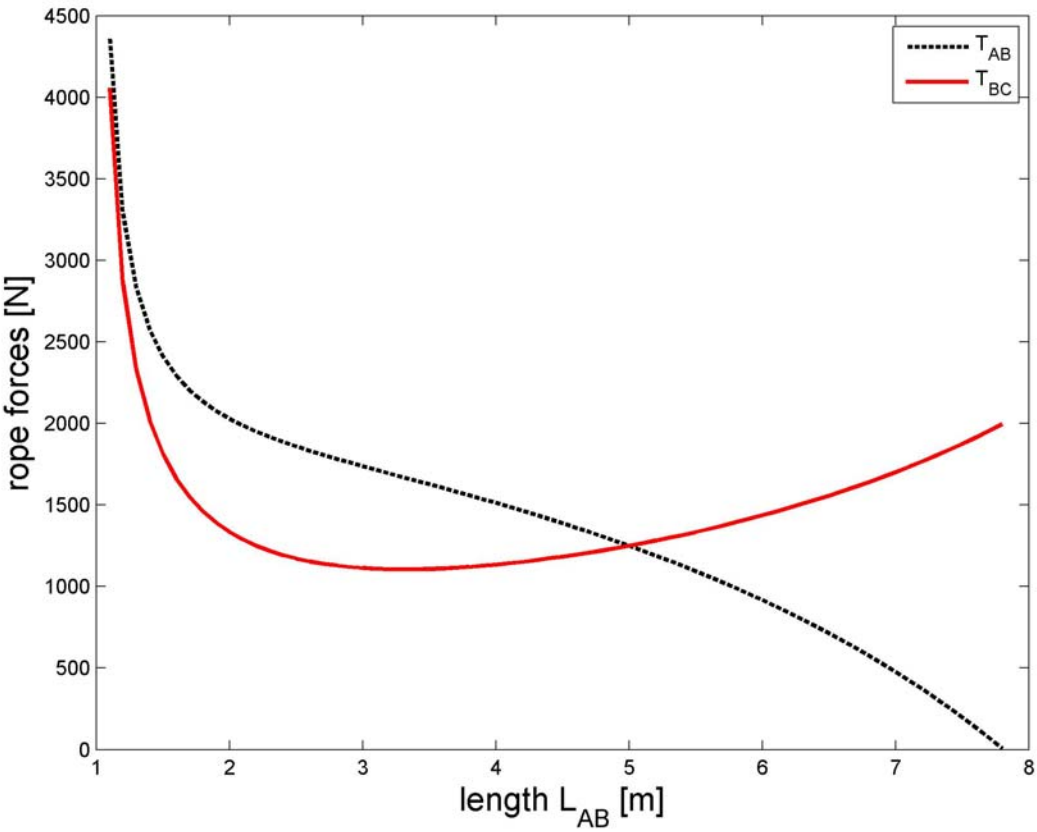


Fig. S40. Rope forces

Discussion

Why the values of forces goes to infinity for $L_{AB} \to 1$?
What length of L_{AB} is required to get a minimum force in the rope BC.
The answer is got from program.

```
Minimum force TBC for a varying length LAB is 1105.5617 [N]
This happens for LAB = 3.3 [m]
```

See the program `S04_ weight_supported_by_two_cables.`

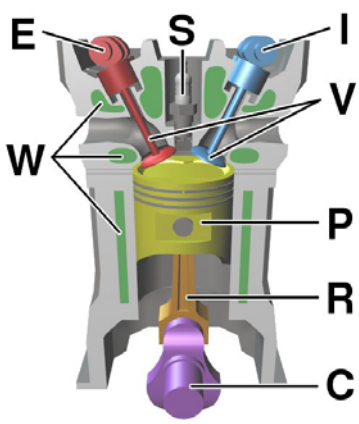
```
% S4_ weight_supported_by_two_cables
% old file name is m_025_weight_supported_by_two_cables_en.m
clear; format short g; format compact
D = 6;
LAB = 3;
LBC = 5;
MG = 2000;
% geometry
theta = acos((D^2 + LAB^2 - LBC^2)/(2*D*LAB));
fi = asin(LAB*sin(theta)/LBC);
% equilibrium
K = [-cos(theta) cos(fi); sin(theta) sin(fi)];
F = [0 MG]';
disp('Rope forces TAB and TBC for given geometry are')
T = K\F;
```

```
disp(T)

LAB_min = D - LBC;
LAB_max = sqrt(D^2 + LBC^2);
incr = 0.1;
% to avoid singularity we start to compute
% the lenght from LAB = LAM_min + incr
LAB_range = LAB_min + incr : incr : LAB_max;
i = 0;
for LAB = LAB_range
    i = i + 1;
    theta = acos((D^2 + LAB^2 - LBC^2)/(2*D*LAB));
    fi = asin(LAB*sin(theta)/LBC);
    K = [-cos(theta) cos(fi); sin(theta) sin(fi)];
    F = [0 MG]';
    T = K\F;
    T_all(i,:) = T;
end

figure(1)
plot(LAB_range,T_all(:,1),'--k',LAB_range,T_all(:,2),'-r','linewidth',2)
legend('T_{AB}', 'T_{BC}'); xlabel('length L_{AB} [m]', 'fontsize', 16);
ylabel('rope forces [N]', 'fontsize', 16)
print -djpeg -r300 f_025_2_en

% find a minimum and its position
[TBC_min i_min] = min(T_all(:,2));
kolik = TBC_min;
disp(['the minimum force TBC for variable length LAB is ' num2str(kolik) ' [N]'])
kde = LAB_range(i_min);
disp(['and occurs for length LAB = ' num2str(kde) ' [m]'])

% end of m_025_weight_supported_by_two_cables_en.m
```

Example – crankshaft mechanism

In Fig. S41 there is schematically depicted a part of the four-stroke engine with its fundamental elements denoted by capital letters. C stands for the crankshaft (crank for short), R for the rod (connecting rod), P for the piston. Other parts, as W – cooling water, E – exhaust cam shaft, I – intake cam shaft, V – intake and exhaust valves and S – spark, are not important for the present analysis. In Fig. S42 the heart of the engine, that is the crankshaft mechanism, is even more simplified.

This is what we call a kinematical scheme of that mechanism. The mechanism has one degree of freedom. We intend to determine the moment M, applied on the rod, which is required to hold the mechanism in its current position against the force P that acts on the piston.



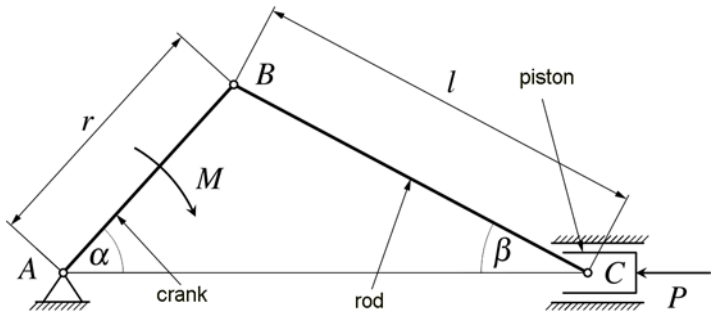


Fig. S41. Four stroke engine Fig. S42. Kinematics scheme of a crankshaft mechanism

Given: The mechanism with one degree of freedom, dimensions, force P.
Determine: The moment M as a function of a constant force P for the crank angle α varying from 0 to 360 degrees. As a parameter consider different values of r/l, that is the ratio of crank radius to the connecting rod length.

The solution is done by solving subsequent equilibrium conditions for individual parts of the mechanism.

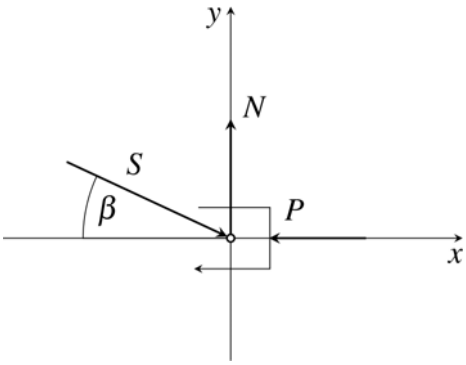


Fig. S43. Free body diagram for a piston

Equilibrium of forces acting on the piston is in Fig. S43.

The force from the rod is S, the normal reaction is N, the force acting on the piston is P. Equilibrium of forces passing through the gudgeon pin at the point C is

$$x: \quad -P + S\cos\beta = 0,$$
$$y: \quad N + S\sin\beta = 0.$$

There are two ways how to express FBD on a piston as shown in Fig. S44. Either as a planar system of forces acting on the body, or as a planar system of forces passing through a point, i.e. the piston pin. The latter approach is a crude simplification whose validity should be properly checked.

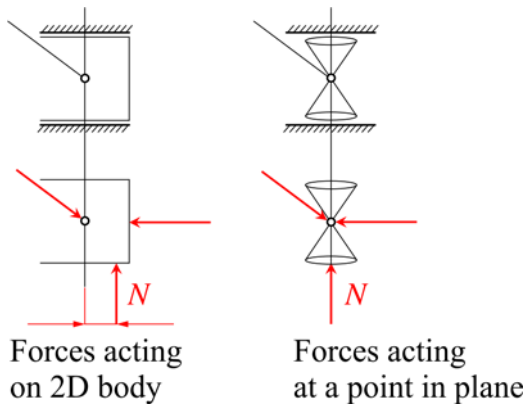


Fig. S44. Piston reactions

Equilibrium of rod forces according to Fig. S45 is

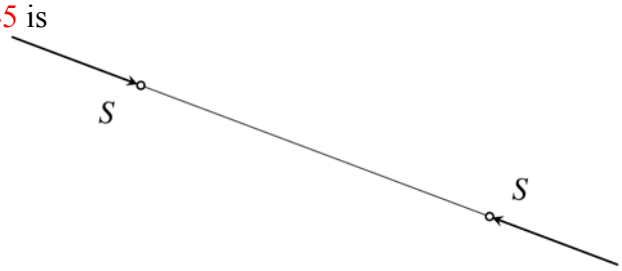


Fig. S45. Rod reactions

Equilibrium of crank forces – Fig. S46.

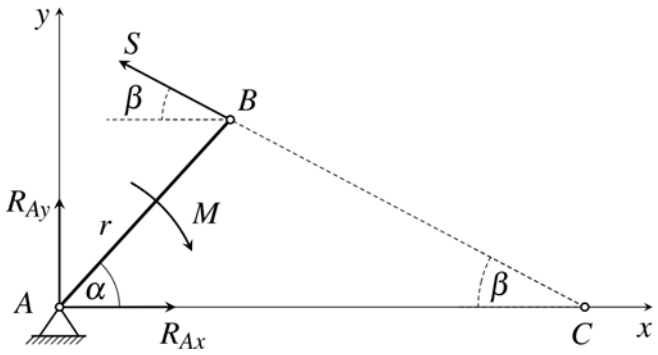


Fig. S46. Crank reactions

To satisfy the equilibrium of planar forces acting on the crank, two component type equations and one moment type equation are needed

$$
\begin{aligned}
x: &\quad R_{\mathrm{AX}} - S\cos\beta = 0,\\
y: &\quad R_{\mathrm{AY}} + S\sin = 0,\\
M_{\mathrm{A}}: &\quad M - Sr\sin\alpha\cos\beta - Sr\cos\alpha\sin\beta = 0.
\end{aligned}
$$

From the third equation, we get the moment acting on the crank

$$M = Sr\,(\sin\alpha\cos\beta - \cos\alpha\sin\beta)\,. \qquad \text{(a)}$$

The system has one degree of freedom, so all coordinates should be expressed as a function of a single variable. For this purpose we have chosen the angle α .

The angle β depends on α by the relation

$$r\sin\alpha = l\sin\beta \quad \Rightarrow \sin\beta = \frac{r}{l}\sin\alpha\,.$$

So the function $\cos\beta$, needed for Eq. (a), could be expressed by

$$\cos\beta = \pm\sqrt{1-\sin^2\beta} = \pm\sqrt{1-\left(\frac{r}{l}\right)^2\sin^2\alpha}$$

or

$$\cos\beta = 1 - 2\sin^2\frac{\beta}{2} = 1 - 2\left(\frac{r}{l}\right)^2\sin^2\frac{\alpha}{2}\,.$$

The latter formula is better, since it does not require a special treatment of $\pm$ sign in front of the square root as it depends on the varying value of α in the range of $\langle 0, 2\pi\rangle$. See the program `S05_crank_shaft_mechanism`.

The problem is solved by the program `S5_crank_shaft_mechanism`.

```
% S05_crank_shaft_mechanism
% old file name is m_005_klikovy_mechanizmus_c3_en.m  (old k6_c4)
% program requires function procedure asin_0_2pi(x)
% crank shaft mechanism - constant force acring on piston
% find dependence of the crank torque on the angular displacement
```

```
% ratio of crank length to rod length is varying
clear
al = 0:pi/16:2*pi;
as = al*180/pi;
p = 1000;   % piston force
r = 0.8;    % crank lenght
l = 2;      % rod length
sinal = sin(al);
cosal = cos(al);
sinal_half = sin(al/2);
r_range = 0.1:0.1:0.9;  %range of r/l
% necessary dimensional requires that
% 2*r must be less than the rod length l
% so rkl must be smaller than 1/2
i = 0;
for r_var = r_range
    i = i + 1; r = r_var; rkl = r/l;
    sinbe = rkl*sinal;
    beta = asin_0_2pi(rkl*sinal);
    % cos(beta) = 1 - 2*sin(beta/2)^2
    beta_half = beta/2;
    cosbe = 1 - 2 * sin(beta_half).^2;
    ss = p./cosbe;
    mm = r*ss.*(cosbe.*sinal + sinbe.*cosal);
figure(1)
subplot(3,3,i)
text = ['r/l = ' num2str(rkl)];
plot(as,mm,'-k', 'linewidth', 3);
title(text, 'fontsize', 16); axis([0 360 -1000 1000]);
grid; xlabel('angular displacement', 'fontsize', 16);
ylabel('torque [Nm]', 'fontsize', 16);
end
print -djpeg -r300 f_005_1_en

%end of m_005_klikovy_mechanizmus_c3_en.m

function x_asin = asin_0_2pi(x)
% vypocti asin(x) v rozsahu 0 az 2*pi
% x musi byt v rozsahu 0 az 2*pi
y = sin(x);
if x<0, x_asin = NaN; end
if (x<=pi/2), x_asin = asin(y); end
if (x<=3*pi/2), x_asin = -asin(y) + pi; end
if (x <= 2*pi), x_asin = asin(y) + 2*pi; end
if (x>2*pi), x_asin = NaN; end
% end of asin_0_2pi.m
```

Fig. S47 shows the rod moment, considering a constant piston force, as a function of the crank angle α for different ratios r/l of the crank radius to the rod length.

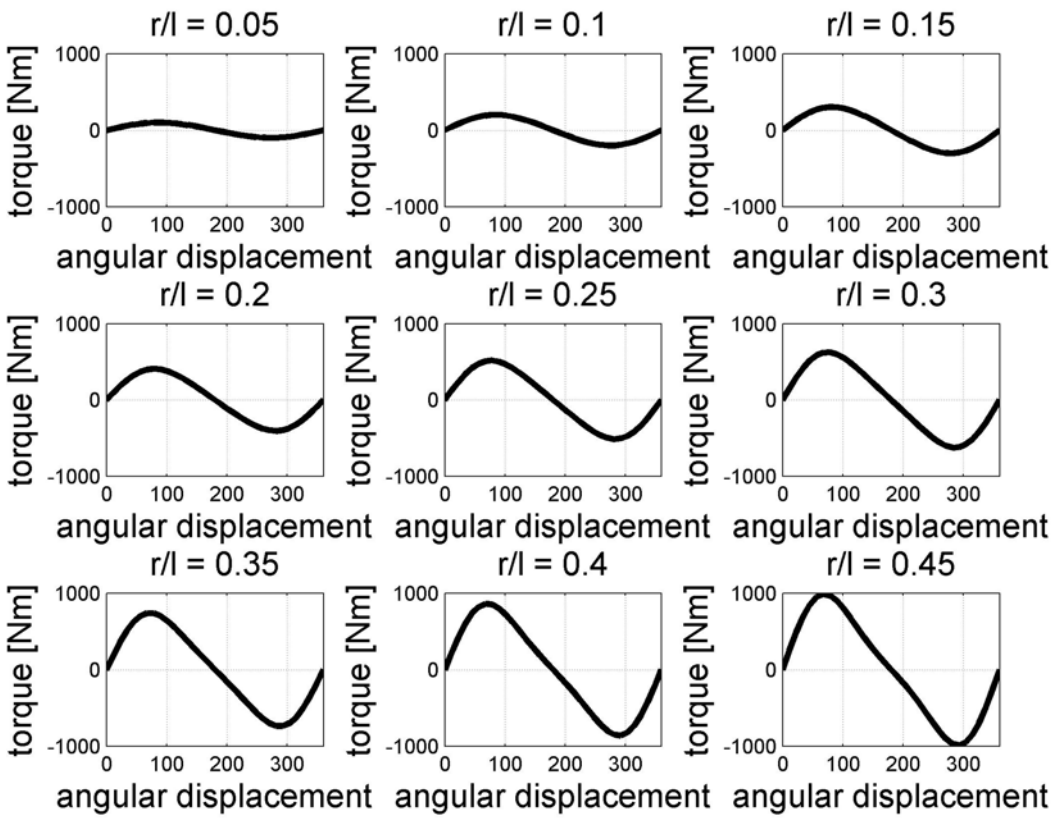


Fig. S47. Torque as a function of angular displacement

Discussion

One can see that a short stroke crank mechanism, with a small ratio of r/l, provides a small moment M with respect to the loading force P, but the function $M = f(\alpha)$ has a rather regular, almost a sine character. The long stroke engine, having a higher ratio r/l, is more 'efficient' – for a given force we get a higher value of the torque M – however, on the expanse of a certain irregularity of the function $M = f(\alpha)$.

The assumption of the constant force P during the rotation of the crank within the range <0, 2π> is not realistic. Actually, one cycle of a four-stroke engine requires four strokes of the piston, that is two complete rotations of the crank, i.e. <0, 4π>. And only one fourth of it corresponds to the expansion part (power stroke) of the cycle. Furthermore, during the expansion part the pressure in the cylinder, i.e. the force acting on the piston, is far from being constant.

The solution presented above could be considered as the first approximation of the task to be refined later on. But this is the way how we generally proceed when analyzing technical problems. Simplifying it as much as possible at first and then gradually taking more and more details into consideration. After all, the real appearance of two parts of crank mechanism, i.e. the connecting rod and piston, shown in Fig. S48, is quite different from the symbolic

representation sketched by a few lines as depicted in the schematic picture Fig. S42. There is a long way from the oversimplified static analysis to an efficient engineering design.

Fig. S48. Piston and connecting rod

Example – simplified truss bridge

The statically determined planar 'bridge' composed of seven rods of equal lengths, connected by five frictionless joints, see Fig. S49, supported by a joint constraint on the left (number 1) and by a rotary sliding joint constraint on the right (number 5), is loaded by a single force Q acting in the lower middle joint (number 3). In the figure the assumed directions of the rod forces are indicated as well.

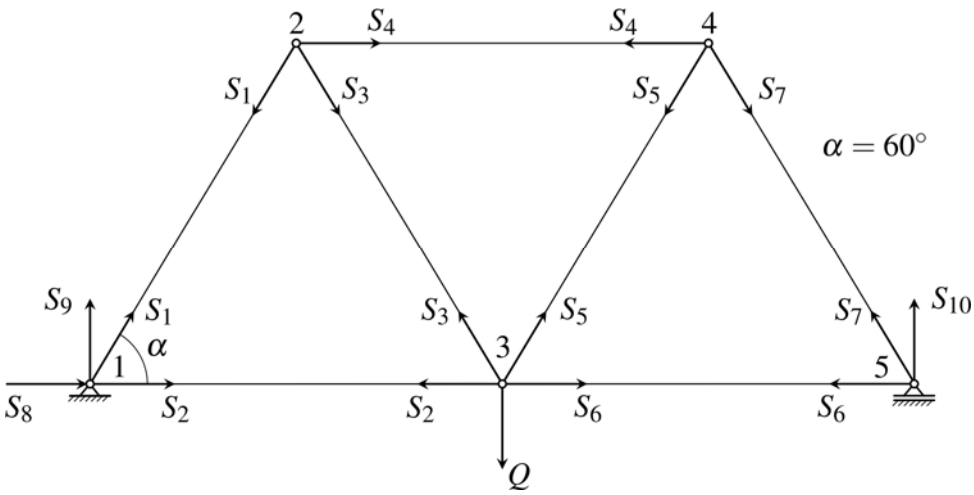


Fig. S49. Truss structure

Given: dimensions, force Q.
Determine: rod forces S_1 *to* S_7 and reactions S_8, S_9, S_{10}.

The task could be solved by expressing equilibrium conditions for individual joints.

Joint 1.
$$S_1 \cos\alpha + S_2 + S_8 = 0,$$
$$S_1 \sin\alpha + S_9 = 0.$$

Joint 2.
$$-S_1 \cos\alpha + S_3 \cos\alpha + S_4 = 0,$$
$$-S_1 \sin\alpha - S_3 \sin\alpha = 0.$$

Joint 3.
$$-S_2 - S_3 \cos\alpha + S_5 \cos\alpha + S_6 = 0,$$
$$S_3 \sin\alpha + S_5 \sin\alpha - Q = 0.$$

Joint 4.
$$-S_4 - S_5 \cos\alpha + S_7 \cos\alpha = 0,$$
$$-S_5 \sin\alpha - S_7 \sin\alpha = 0.$$

Joint 5. $$-S_6 - S_7 \cos\alpha = 0,$$
$$S_7 \sin\alpha + S_{10} = 0.$$

Altogether we have ten equations for ten unknown reaction forces. The following program `S06_truss_bridge` shows how to proceed in Matlab.

```
% S06_truss_bridge
% prutovka_stat_urcita_silova_metoda_c1
clear
al = pi/3;
sn = sin(al);
cs = cos(al);
% matrix [K]
K = [ cs  1 0     0   0   0   0 1 0 0;
      sn  0 0     0   0   0   0 0 1 0;
     -cs  0 cs    1   0   0   0 0 0 0;
     -sn  0 -sn   0   0   0   0 0 0 0;
       0 -1 -cs   0  cs   1   0 0 0 0;
       0  0  sn   0  sn   0   0 0 0 0;
       0  0   0  -1 -cs   0  cs 0 0 0;
       0  0   0   0 -sn   0 -sn 0 0 0;
       0  0   0   0   0  -1 -cs 0 0 0;
       0  0   0   0   0   0  sn 0 0 1];
 % loading forces
 F = zeros(10,1);
 Q = 1000;
 F(6) = Q;
 F;
 % solving the equilibrium conditions we get
 S = K\F
```

The rod forces ($S_1 \cdots S_7$) and reaction forces (S_8, S_9, S_{10}) are

```
 1 -577.3503
 2  288.6751
 3  577.3503
 4 -577.3503
 5  577.3503
 6  288.6751
 7 -577.3503
 8    0.0
 9  500.0000
10  500.0000
```

The reader should check the equilibrium conditions. How?

Discussion

If the ‘bridge’ were supported on the right the same way as it is on the left-hand side, then the number of dof’s would be equal to -1. Such a system would be classified as interdetemined. The unknown variables could not be determined since their number (4 + 7 = 11 in this case) is greater than the number of available equations (still ten only). Later, we will show how the tasks of this type are solved by tools of mechanics of deformable bodies.

Hint – plot truss bridge in Matlab

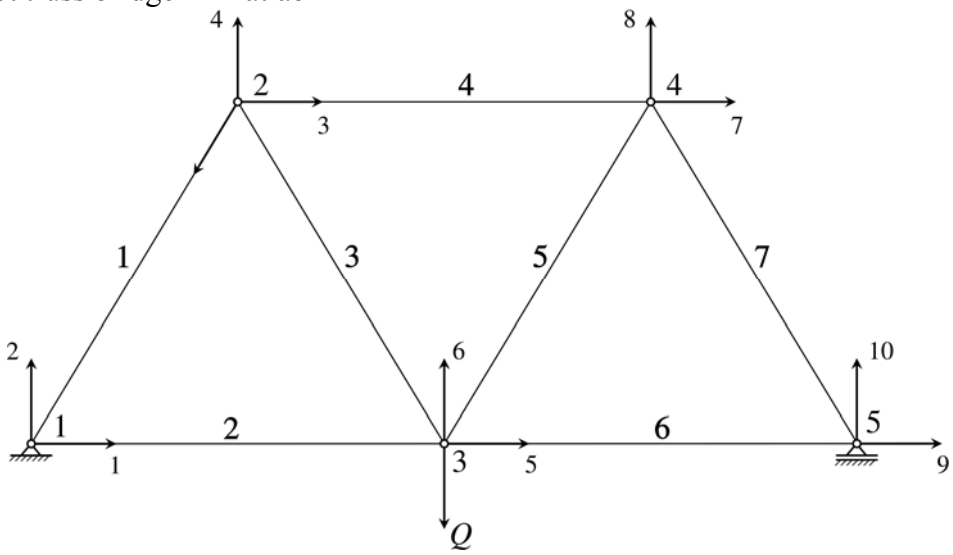


Fig. S50. Nodes, trusses, displacements

Matlab could help when the bridge structure, depicted in Fig S50, have to be plotted. All the rods are of the same length. The coordinates of all joints (stored in the array `xy`) are given. The following piece of program shows how to proceed using the Matlab function `gplot(C,xy)`. The array `C`, called the connectivity matrix, indicates the nodes that have to be connected by a line. See the Matlab program `S07_plot_a_truss_structure` and Fig. S51.

```
% S07_plot_a_truss_structure
% old file name is m_008_nakresli_prutovou soustavu.m

clear
% geometrie
l = 1;
alfa = pi/3; % 60 degrees
ly = l*sin(alfa);
% nodal coordinates
xy(1,:) = [0 0];
xy(2,:) = [l/2 ly];
xy(3,:) = [1 0];
xy(4,:) = [3/2 ly];
xy(5,:) = [2,0];
% conectivity
C(1,2) = 1;
C(1,3) = 1;
C(2,3) = 1;
C(2,4) = 1;
C(3,4) = 1;
C(3,5) = 1;
C(4,5) = 1;
figure(1)
subplot(2,1,1); spy(C);
title('spy(C)', 'fontsize', 16)
subplot(2,1,2); gplot(C,xy);
axis('equal'); axis([-0.1 2.1 -0.1 1])
title('gplot(C,xy)', 'fontsize', 16)
hold on          % hold for a moment
for i = 1:5      % and plot the nodes
    plot(xy(i,1),xy(i,2),'o', 'linewidth', 2)
end
hold off         % its all
print -djpeg -r300 f_008_2

% end of m_008_nakresli_prutovou soustavu.m
```

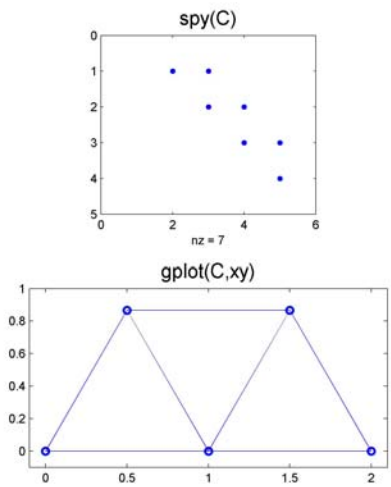


Fig. S51. Matlab plot

Example – movable system with one dof

Type of task: find the equilibrium configuration for a movable system with one dof.
Given: In Fig. S52 there is a planar mechanism with one degree of freedom. It consists of a sleeve, having weight Q, which could move in up or down directions along a vertical rod attached to the frame. At the point A of the sleeve there is attached a rope that is led around the pulley that could rotate around the frictionless joint at point S. The other end of the rope is loaded by a force Z. Friction effects are neglected.

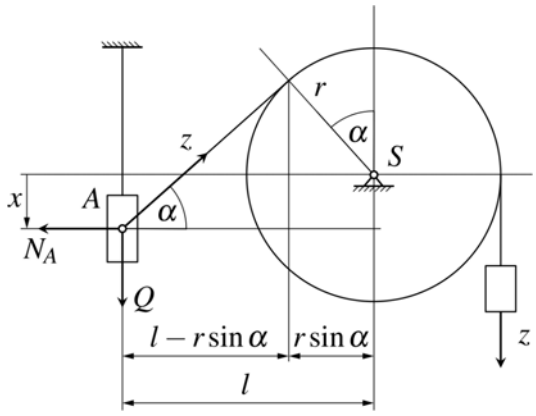


Fig. S52. Pulley and sleeve equilibrium

Determine:
For given loads (Q and Z) and for given dimensions find the configuration of the mechanism, determined by the coordinate x, in which the equilibrium occurs.

One way to solve the problem is to analyze equilibrium of forces at point A. Two component type equations are

$$-N_A + Z\cos\alpha = 0,$$
$$-Q + Z\sin\alpha = 0.$$
$$----------$$
$$\Rightarrow \sin\alpha = \frac{Q}{Z}, \quad \alpha = \arcsin\frac{Q}{Z}.$$

From geometry considerations, we get the distance x, oriented downwards, as a function of the angle α from

$$\tan\alpha = \frac{x + r\cos\alpha}{l - r\sin\alpha} \Rightarrow x = -r\cos\alpha + (l - r\sin\alpha)\tan\alpha\,.$$

This way the reactions at joint S are not obtained. But nobody asked for it so far. Show, however, how the task might be solved if the reaction force at the joint S is required.

Discussion
Since the function arcsin appears in the analysis, there is a natural limit for its argument. So, it is necessary that $-1 < Q/Z < 1$. The physical, or rather the geometrical, meaning of this fact is that as the ratio Q/Z approaches to $+1$ or -1, then the distance x goes beyond all limits. So, there is an embedded singularity in the solution that might be described by

$$\lim_{Q/Z\to\pm1} x = \lim_{Q/Z\to\pm1}\left[-r\cos(\arcsin\frac{Q}{Z}) + (l - r\sin(\arcsin\frac{Q}{Z}))\tan(\arcsin\frac{Q}{Z})\right] \to \pm\infty\,.$$

See the Matlab program `S08_sleeve_and_pulley.m`. The output is in Fig. S53.

```
% S08_sleeve_and_pulley
C:\tmp_matlab_2016\rest_123
% test_123
clear
l=2; r = 1;
Q = -9.9:0.1:9.9; Z = 10;
alfa = asin(Q/Z);

x = -r*cos(alfa) + (l - r*sin(alfa)).*tan(alfa);
figure(1)
subplot(1,3,1); plot(Q,x, 'linewidth', 2); grid; xlabel('Q'); ylabel('x'); ...
    title('x = f(Q)')
subplot(1,3,2); plot(Q,alfa*180/pi, 'linewidth', 2); axis([-10 10 -90 90]); ...
    grid; xlabel('Q'); ylabel('\alpha'); title('\alpha = f(Q)')
subplot(1,3,3); plot(Q/Z,alfa*180/pi, 'linewidth', 2); xlabel('Q/Z');  ...
    ylabel('\alpha'); grid; axis([-1 1 -90 90]); title('\alpha = f(Q/Z)')

figure(2)
subplot(1,2,1); plot(Q/Z,x, 'linewidth', 2); grid; xlabel('Q/Z [1]', 'fontsize' ,16); ...
    ylabel('x [m]', 'fontsize' ,16); title('x = f_1(Q/Z)', 'fontsize' ,16)
subplot(1,2,2); plot(Q/Z,alfa*180/pi, 'linewidth', 2); xlabel('Q/Z [1]', 'fontsize' ,16);  ...
    ylabel('\alpha [degrees]', 'fontsize' ,16); grid; axis([-1 1 -90 90]); ...
    title('\alpha = f_2(Q/Z)', 'fontsize' ,16)

print -djpeg -r300 rest123_fig_2
```

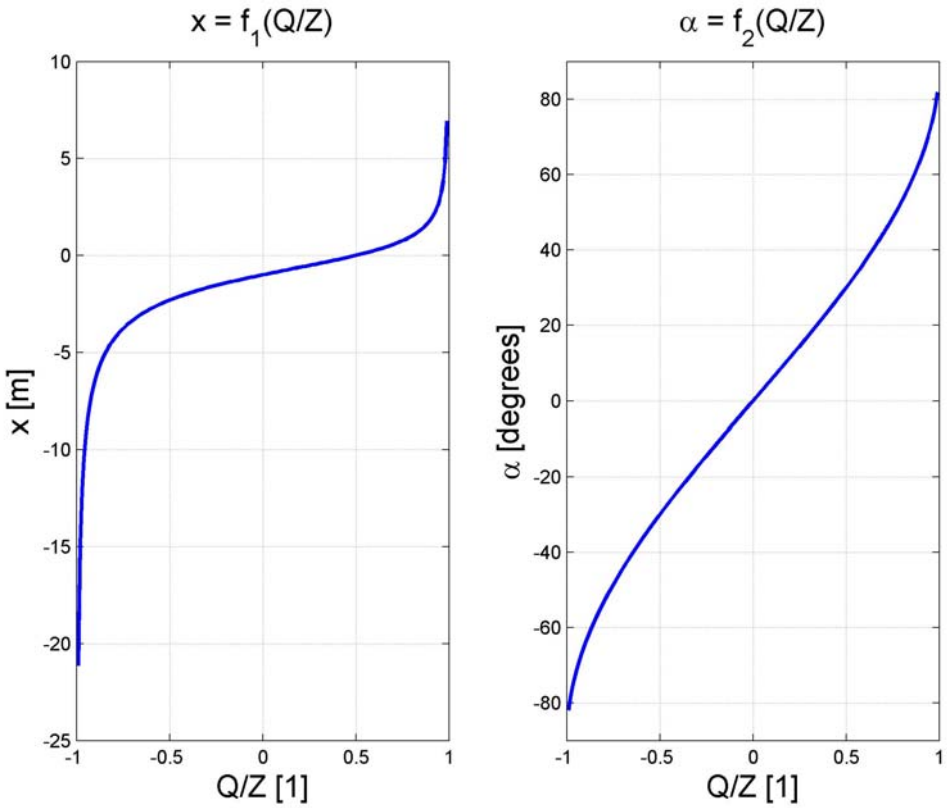


Fig. S53. Matlab output

Example – system of two connected bodies in plane

Type of task: zero dof's.
Given: Dimensions, forces P_2, P_3.
Determine: All the reactions.

The structure, composed of a rotary bar (3) and a slider (2), connected in a frictionless joint A, is depicted in Fig. S54. The corresponding FBD is in Fig S55.

Equations of equilibrium are

Body 2

$x: \quad A_x - P_2 = 0,$

$y: \quad -A_y + N = 0,$

$M_A: \quad P_2 q - Nz = 0.$

Body 3

$x: \quad -A_x + R_{Bx} = 0,$

$y: \quad A_y - P_3 + R_{By} = 0,$

$M_B: \quad -P_3 p \sin\alpha + A_x l \cos\alpha + A_y l \sin\alpha = 0.$

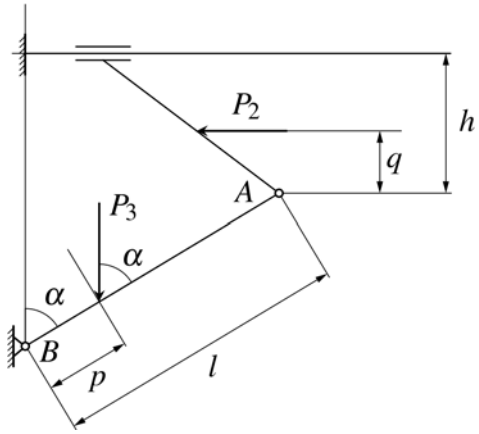


Fig. S54. Two bodies

There are six equations for six unknowns, i.e. for $A_x, A_y, R_{Bx}, R_{By}, N, x$.

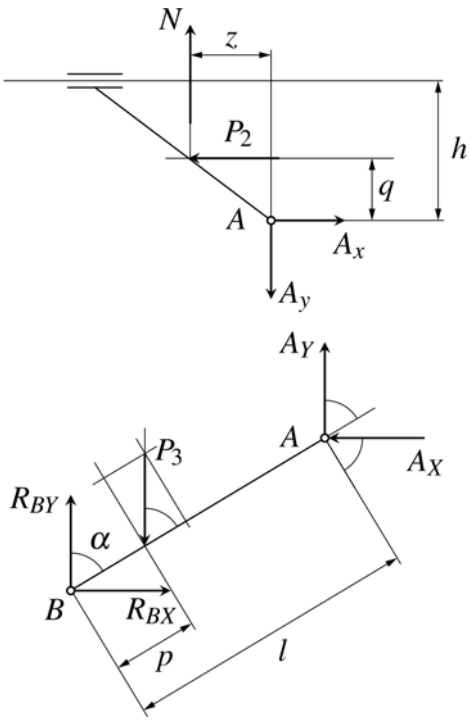
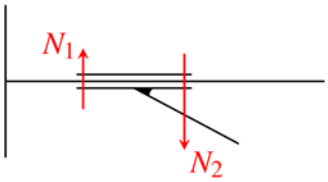


Fig. S55. Free body diagrams

Hint – why the normal reaction appears to be out of the sleeve

Explain, why the normal reaction between the sleeve with a handle and the rod along which it slides, is seemingly out of the contact area as it is indicated in the third subplot of Fig. S56.

To secure a smooth motion of the sleeve along the rod, there has to be a certain radial gap. When the handle of a sleeve is loaded, then the sleeve tilts a little bit and the actual contact occurs at the side parts of the sleeve as it is shown in the first two subplots of Fig. S56.



The reaction forces N_1, N_2 between the collar and the sleeve are parallel, perpendicular to the rod, and generally of different magnitudes. And the resulting force N, being the vector sum of N_1, N_2, always occurs out of the centre of the sleeve. So, when plotting FBD one can use either two unequal forces N_1, N_2 or just a single force N displaced by an unknown distance x. In both cases, the number of unknowns is two.

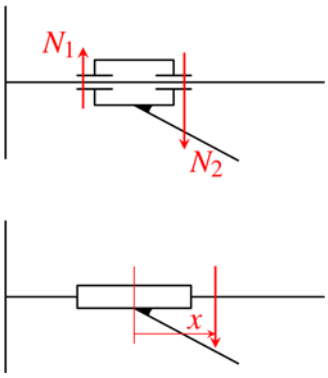


Fig. S56. Sleeve reactions – alternatives

In Fig. S57 a task of finding the force N, being the resultant of N_1, N_2, is shown both by graphic and analytical approaches.

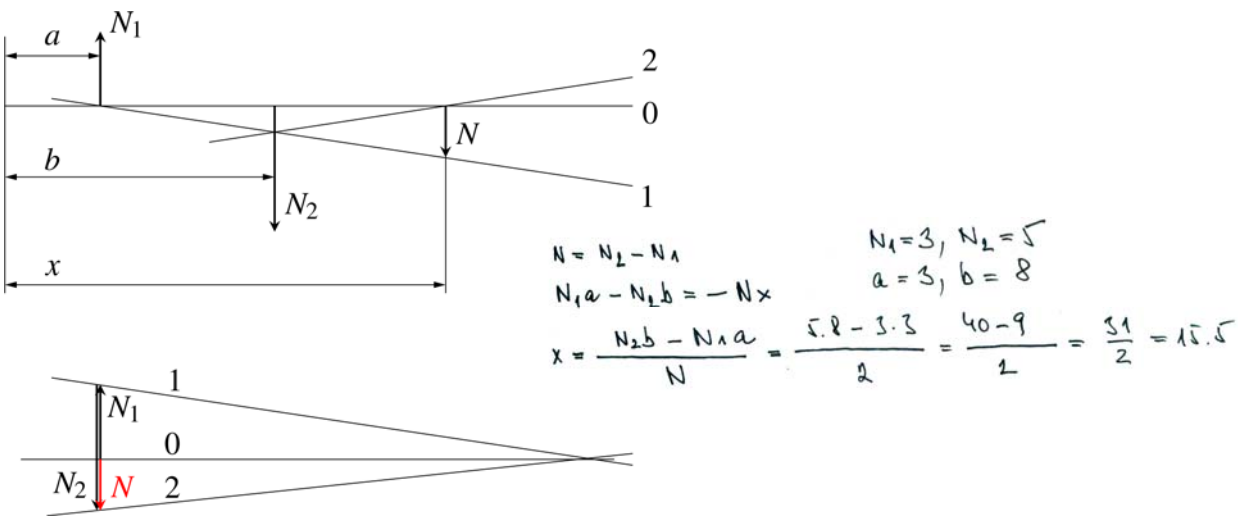


Fig. S57. Parallel forces – graphical and numerical approach

S9. Friction

Friction is a phenomenon appearing when surfaces of bodies in contact are in relative motions. Friction induces forces acting against the motion. The behavior of mechanical systems is always accompanied by the occurrence of effects resisting motions, or by other words, of forces (and moments) acting against the motion.

These effects are of different nature as the dry friction, fluid friction, internal friction, etc. The common property of frictional effects is that they irreversibly dissipate energy.

In the text, we will devote our attention to the phenomenon of *dry friction*, frequently occurring in contact surfaces of bodies. The actual contact surface is often approximated by a point.

The mathematical description of dry friction is a subject of tribology and is far from being simple[9]. For our purposes, a simplifying so-called *phenomenological approach*, known as the *Coulomb's law*[10], will be used.

There are two distinct regimes of dry friction; they are called kinetic and static frictions, respectively.

S9.1. Kinetic friction, frequently called just **friction**, is defined for sliding bodies. The friction force is approximated by the formula

$$F = Nf \ , \qquad \text{(S9\_1)}$$

where

F is the friction force acting in the contact of sliding bodies. The force lies in the tangent plane between the contact surfaces of bodies and its direction is opposite to relative velocities of contact surfaces.

N is the normal reaction in the contact of sliding bodies and

f is the *coefficient of friction*. Its value, depending on the type of contacting surfaces, can be found in engineering handbooks. Often, the friction coefficient is denoted by the symbol μ.

Expressed in words, the Coulomb's law states that the **friction force** is proportional to the normal force in contact.

[9] *Neale, Michael J. (1995). The Tribology Handbook (2nd Edition). Elsevier.* ISBN 9780750611985.
[10] One has to realize that the Coulomb law is an approximation of real world assuming that the friction phenomena are independent of the sliding velocity, magnitude of normal force, temperature, humidity, surface structure, etc.

A few examples of values of coefficient friction f for different sliding surfaces

Steel – ice	0.02
Steel – steel	0.15
Steel – stone	0.30
Steel – sand	0.40

It is a dimensionless quantity, whose value is obtained experimentally. For more details see www.engineershandbook.com/Tables/frictioncoefficients.htm.

S9.2. Static friction – also called **adhesion** – defined for bodies in contact that are not moving relative to each other, is approximated by

$$F \le N f_{\mathrm{a}}, \qquad \text{(S9_2)}$$

where

F is the adhesion force acting in the contact between stationary bodies. The force lies in the tangent plane between the contact surfaces and its direction is a priory unknown.

The force N is the normal reaction in the contact between the bodies and

f_{a} is the dimensionless *adhesion coefficient*. Its value, depending on the type of contacting materials can be found in tables of engineering textbooks.

The adhesion force can take any value within the interval $\langle -Nf_{\mathrm{a}}, +Nf_{\mathrm{a}} \rangle$.

Expressed in words, the **adhesion force** is just what it must be in order to prevent motion between the surfaces of contacting bodies.

The adhesion coefficient is usually higher than the coefficient of kinetic friction.

In left-hand side of Fig. S58 there are shown reaction forces for a general planar constraint contact taking friction phenomenon into account. It is assumed that the upper 'body' moves to the right with velocity v. Besides of the normal reaction N, which is perpendicular to the mutual tangent to both surfaces, there is the friction force, lying in the tangent line and having a direction opposite to the relative motion of surfaces.

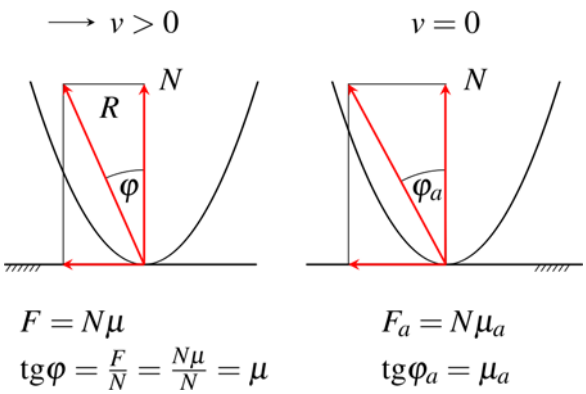


Fig. S58. Friction – adhesion

According to Coulomb's law, its magnitude is proportional to the normal force N, while the coefficient of proportionality is just the coefficient of friction f.

The resulting reaction is obtained as a vector sum of both vectors, i.e. $R = \sqrt{N^2 + F^2}$. This sliding constraint represents one unknown reaction component, i.e. the normal reaction. It is of interest that the angle φ, sometimes called the *friction angle*, can be obtained from

$$\tan\varphi = \frac{F}{N} = \frac{Nf}{N} = f \text{ , so } \varphi = \arctan f \text{ .} \qquad \text{(S9\_3)}$$

In the right-hand side of Fig. S58 there are shown reaction forces for a general planar constraint contact taking friction phenomenon into account. Now, it is assumed that both surfaces are stationary, i.e. $v = 0$. It should be emphasized that in this case, the direction of the actual adhesion force is unknown (it could point either to the left or to the right) and the magnitude of the adhesion force unknown as well. This stationary constraint represents two unknown reaction components, i.e. the normal reaction N and the adhesion force F_a. The adhesion angle is $\varphi_a = \arctan f_a$.

S9.3. Normal and friction forces in a contact between extended surfaces

If a loaded block, shown in Fig. S59 moves to the right, one might be wondering where the normal force, acting between the frame and the block, should occur. Actually, the normal force, we intend to plot in the FBD, is a resultant of generally nonuniform contact pressure (of course, multiplied by the magnitude of the surface area) between the block and the frame.

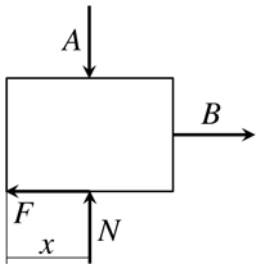


Fig. S59. Position of a normal reaction

The distribution of the pressure along the contact surface is a priory unknown since it depends on the actual loading of the block. Since we are only interested in the resultant value, say N , we might assume that it is located at an unknown distance x from the left-hand side of the block. The quantities N, x appearing in FBD are unknown.

The friction force, being by definition $F = Nf$, lies in the contact 'plane' and its lateral position is immaterial. So, when solving the task to find a force B, needed to pull the block with a constant velocity to the right, and knowing the force A and the coefficient of friction f , we have to write three equations for the block – two component and one moment type equilibrium equations – which would allow to find three unknowns, i.e. B, N and x .

Example – forces acting on a 2D block in plane. See Fig. S60.
Given: The block of weight Q lies on an inclined plane and is also supported by a pin constraint at the point A. It is loaded by forces Z_1, Z_2 .
Determine: Reactions.

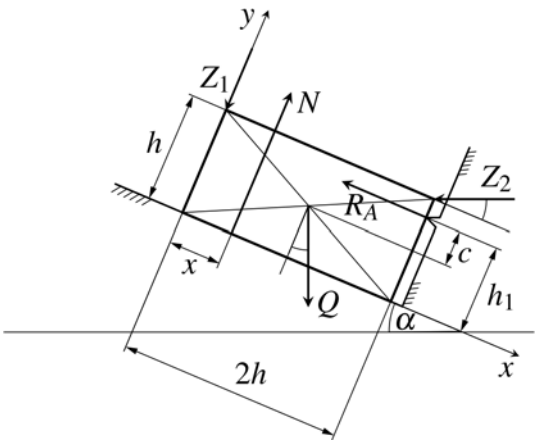


Fig. S60. A loaded block

Three equations are required to express equilibrium conditions of a body in 2D space.

$$
\begin{aligned}
x: &\quad -R_A - Z_2\cos\alpha + Q\sin\alpha = 0,\\
y: &\quad N - Z_1 - Z_2\sin\alpha - Q\cos\alpha = 0,\\
M_A: &\quad 2lZ_1 + Qc\sin\alpha + Ql\cos\alpha - N(2l - x) + Z_2(h - h_1)\cos\alpha = 0.
\end{aligned}
$$

$\Rightarrow R_A, N, x.$

S9.4. Normal and traction forces in a pure rolling contact

In Fig. S61 there is depicted a driven (or braked) round wheel on an inclined plane in 2D. Also, the FBD forces and moments are indicated. The condition of pure rolling requires that there is no slipping between the wheel and the supporting frame. The corresponding constraint force, called the traction force F, has to be smaller than Nf. Writing three equilibrium equations for a 2D body and knowing $mg_{\text{wheel}}, F_x, F_y$, one can evaluate three unknowns i.e. M, F, N, needed for the wheel to move with a constant velocity.

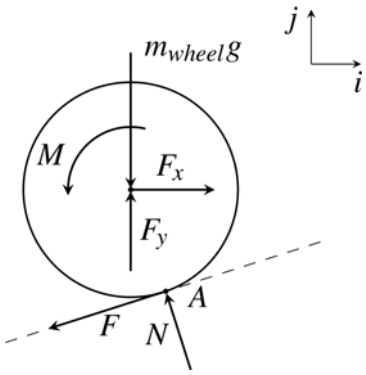


Fig. S61. Rolling contact

We have to check whether the condition of rolling, i.e. $F < Nf$, is satisfied. If this condition is not satisfied, it means that the initial assumption of pure rolling was wrong. The task has to be recomputed under the assumption of slipping, i.e. $F = Nf$.

S10. Rolling resistance

Even if we are dealing with mechanics of rigid (non-deformable) bodies, the phenomenon of the rolling resistance can be best explained by a logical sidestep. Imagine that a loaded rigid wheel is rolling on a slightly deformable surface (frame). See Fig. S62.

Due to the deformed frame the normal reaction N is shifted slightly (from the ideal contact point) by the distance ξ to the right. The traction force, which has to be smaller than Nf, is directed against the motion. The resulting reaction is R. To simplify the analysis of the task and the plotting of the FBD we usually shift the normal force to the ideal contact point P.

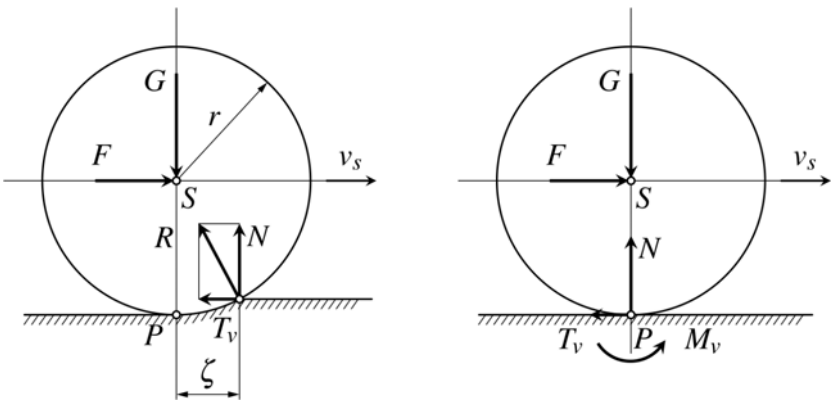


Fig. S62. Rolling resistance

This artificial shift has to be accompanied by a corresponding moment M_v, whose magnitude is $N\xi$.

The coefficient ξ goes under the name of the *coefficient of rolling resistance.* Its value for different contacting surfaces can be found in engineering textbooks.

S11. Principle of virtual work (PVW)

The *virtual work* is mechanical work produced by forces exerted during their virtual displacements. By the term *virtual displacement* we understand any infinitesimal displacements and rotations, satisfying the prescribed constraint conditions. For virtual quantities Lagrange introduced the symbol δ, to emphasize the virtual, i.e. the fictional or apparent, character of these quantities. We assume that while the body is being transferred to a new, infinitesimally close position, the acting forces do not change their magnitudes and directions and simultaneously that the time is frozen. The difference between the virtual and differential quantity can be explained observing Fig. S63.

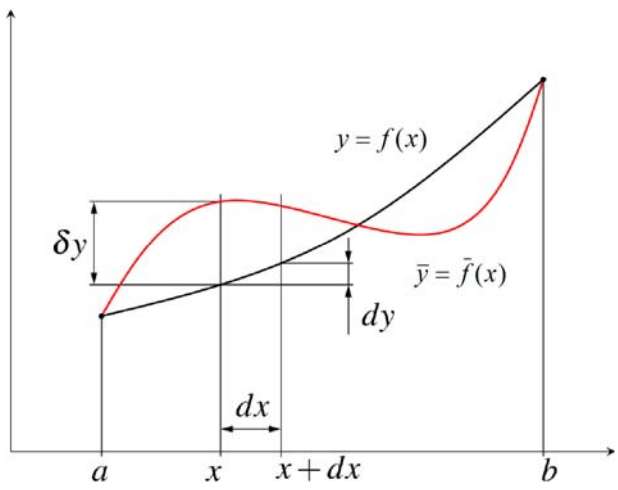


Fig. S63. Variation vs. derivative

Let the function $y = f(x)$ represents the relation between two quantities, say the displacement and time. Let's have another function $\bar{y} = \bar{f}(x)$ and let it be defined as the virtual variation of the original function. According to rules of infinitesimal calculus, there is a unique correspondence between differential increments $\mathrm{d}x$ and $\mathrm{d}y$ depending on the function $f(x)$. Contrary to the differential increment $\mathrm{d}y$, the virtual increment is defined as $\delta y = \bar{y} - y$. More about the subject can be found in books devoted to variational calculus. See [4].

In mechanics of deformable bodies (strength of material) the principle of virtual work states that the virtual work of internal forces , say δU, is equal to the virtual work of external forces, say δW , so

$$\delta W = \delta U \,. \qquad \text{(S11\_1)}$$

In mechanics of rigid bodies the deformations of loaded bodies are neglected, so the work done by internal forces is assumed to be identically equal to zero, thus

$$\delta W = 0 \,. \qquad \text{(S11\_2)}$$

It can be proved that zero work of forces acting during the virtual displacement corresponds to the equilibrium condition stating that the sum of forces and moments acting on a body is equal to zero.

At the first sight the conclusion, that the zero resulting force produces zero work, seems to be trivial. But, the resulting zero is a sum of non-zero contributions of works produced by virtual displacements of individual forces. We will show that the strength of the principle is based on the fact that it has to be valid for an arbitrary virtual displacement.

When balancing individual work contributions we rely on the fact that in mechanics of rigid bodies the internal forces – when the resistance effects are neglected – 'do not work'. Furthermore, according to action and reaction principle, they are equal but of different directions. The principle allows advantageous solving static tasks without the necessity to evaluate all the reaction forces. The principle of virtual work loses its simplicity when resistance forces are taken into account.

Example – work done by a force acting on a spring

Given: A linear spring with stiffness k is gradually loaded by the force P. See Fig. S64. Its magnitude is linearly increasing from zero to the maximum value $P_{\max}$. The deflection is proportional to the applied force, thus $P = ky$, where k is the spring stiffness. Consequently, the spring deflection y goes from zero to $y_{\max}$.

Determine: The work W exerted by the applied force P during the loading process.

$$W = \int_0^{y_{\max}} P\,\mathrm{d}y = \int_0^{y_{\max}} ky\,\mathrm{d}y = \frac{1}{2}ky_{\max}^2 .$$

We could express the stiffness as $k = \dfrac{P_{\max}}{y_{\max}}$ and substitute it into the previous equation and obtain

$$W = \frac{1}{2}P_{\max}\,y_{\max} .$$

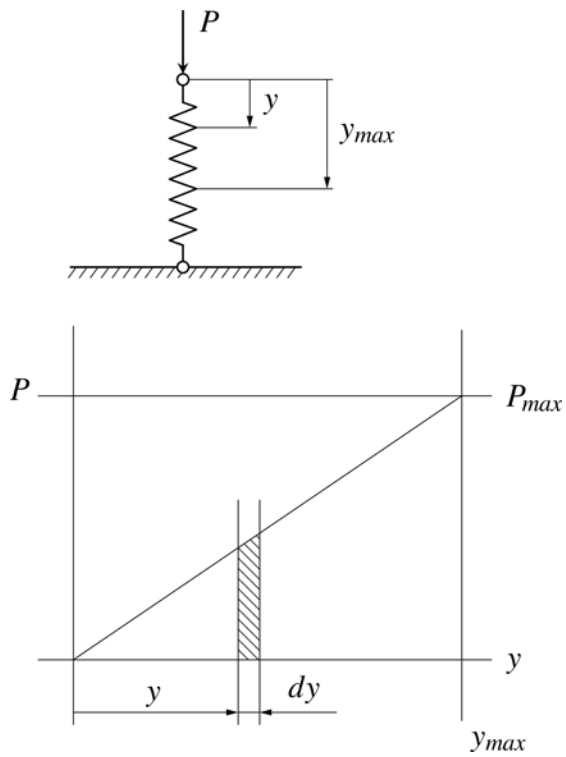


Fig. S64. Work done by a constant force acting on a linear spring

Example – crankshaft mechanism

Type of task: mechanism with 1 dof , no friction considered.
Given: dimensions, force P , see Fig. S65.
Determine: moment M needed for mechanism to stay in the shown configuration using PVW.

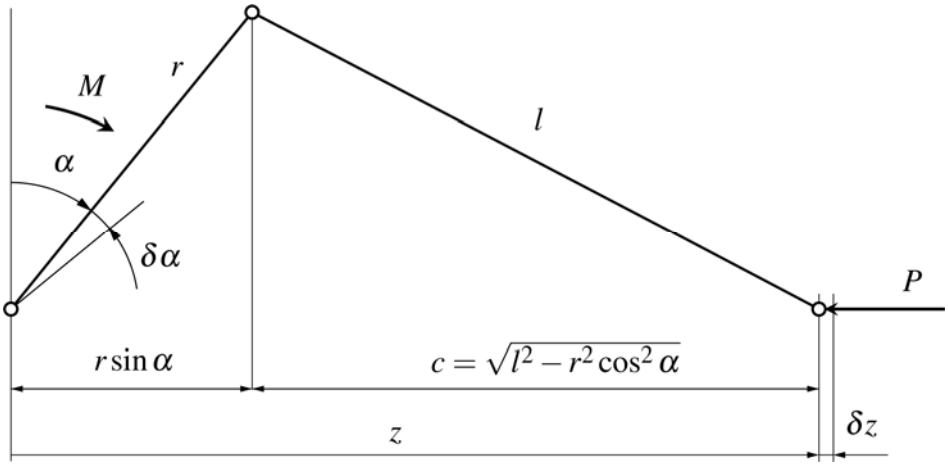


Fig. S65. Principle of virtual work applied to a crankshaft mechanism

The principle of virtual work requires that for an infinitesimal change of the current position of the mechanism, the sum of the virtual work of the moment M and of the virtual work of the force P has to be zero. Since the mechanism has just one dof, there exists a single coordinate uniquely determining its position. Opting for the crank angular coordinate, say α , as the primary coordinate, the piston displacement z depends on it and has to be expressed as a function of α . Similarly for δz .

Observing Fig. S65 one can write

$$z = r\sin\alpha + c\,.$$

Using Pythagoras theorem gives

$$r^2\cos^2\alpha + c^2 = l^2 \qquad \Rightarrow \quad c = \pm\sqrt{l^2 - r^2\cos^2\alpha}\,.$$

So, the piston position depends on the crank angle α by

$$z = r\sin\alpha + \sqrt{l^2 - r^2 s\cos^2\alpha}\,. \tag{a}$$

Considering the clockwise orientation of the angle as α positive, the virtual increment $\delta\alpha$ is positive in the clockwise direction as well. One can observe that increasing angle α by a positive increment $\delta\alpha$ leads to an increase of z by a positive value of δz. So, the principle of virtual work states that

$$\delta W = M\,\delta\alpha - P\,\delta z = 0\,. \tag{b}$$

Note: The minus sign by the second term is due to the fact that the force P, as it is plotted in FBD, acts against the positive virtual increment of δz.

The relation between $\delta z, \delta\alpha$ can be found by differentiating Eq. (a).

$$\frac{\delta z}{\delta\alpha} = r\cos\alpha + \frac{-2r^2\cos\alpha(-\sin\alpha)}{2\sqrt{l^2 - r^2\cos^2\alpha}}. \tag{c}$$

This can also be done by the program `S09_crank_virt_work_c1.m`

```
% S09_crank_virt_work_c1
% original file name is crank_virt_work_c1
clear
syms r l alfa z z1 z2
z1 = r*sin(alfa);
z2 = sqrt(l^2 - r^2*cos(alfa)^2);
z = z1 + z2;
dz1_to_dalfa = diff(z1,alfa);
dz2_to_dalfa = diff(z2,alfa);
dz_to_dalfa = dz1_to_dalfa + dz2_to_dalfa;
pretty(dz_to_dalfa)
```

Executing it we get

```
                    2
                   r  cos(alfa) sin(alfa)
r cos(alfa) + ----------------------.
                  2    2          2 1/2
                 (l  - r  cos(alfa) )
```

And similarly for virtual increments

$$\frac{\delta z}{\delta\alpha} = r\cos\alpha + \frac{-2r^2\cos\alpha(-\sin\alpha)}{2\sqrt{l^2 - r^2\cos^2\alpha}}.$$

The virtual displacement of the piston ∂z depends on the virtual rotational increment $\delta\alpha$ by

$$\delta z = \left[-r\cos\alpha - \frac{r^2\sin\alpha\cos\alpha}{\sqrt{l^2 - r^2\cos^2\alpha}}\right]\delta\alpha.$$

Substituting into (b) and factoring out $\partial\alpha$ we get

$$\delta\alpha\left\{M - P\left[r\cos\alpha + \frac{r^2\sin\alpha\cos\alpha}{\sqrt{l^2 - r^2\cos^2\alpha}}\right]\right\} = 0. \tag{d}$$

And now, comes the most important logical step for the understanding the principle of virtual work. The last relation consists of a product of two terms that are equal to zero. That is the virtual displacement $\delta\alpha$ and the rest of the relation, which is contained in braces. The mathematical condition for the product (d) to be equal to zero for any value of $\delta\alpha$ requires that the contents of the bracket has to be equal to zero. And what is inside the braces corresponds to equilibrium condition.

Thus

$$M = P\left[r\cos\alpha + \frac{r^2 \sin\alpha\cos\alpha}{\sqrt{l^2 - r^2\cos^2\alpha}}\right].$$

This way we found the required relationship between the forces needed for the equilibrium of the mechanism without a necessity to determine reactions and internal forces. The beauty and the simplicity of the task would be lost if passive resistance effects were taken into account.

Example – compare FBD and PVW solutions

Type of task: 1dof system.
Given: Dimensions. A sleeve of the weight G can move up and down along a vertical frictionless rod. The sleeve is also attached to the frame by a linear massless spring whose initial (unstretched) lenght l_0 is equal to b and its stiffness is c. See Fig. S66.

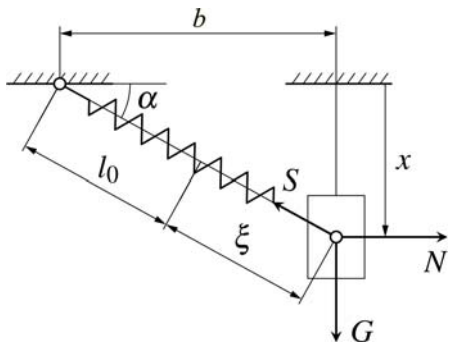


Fig. S66. Sleeve and spring equilibrium

Determine: For the given weight G find the equilibrium position indicated by x coordinate. Compare the classical solution, obtained by FBD technique, and that obtained by the principle of virtual work.

1) FBD solution

The condition of equilibrium of forces passing through a single point in a plane requires writing two equations.

$$\begin{aligned} -S\cos\alpha + N &= 0, \\ -S\sin\alpha + G &= 0. \end{aligned} \qquad \text{(a,b)}$$

Since the spring is assumed linear, then the force in the spring S is proportional to the elongation ξ, and thus the constitutive equation is

$$S = c\xi, \qquad \text{(c)}$$

where c is the stiffness of the spring.

The relations between the distance x, elongation ξ and the angle α come from geometry.

$$\sin\alpha = \frac{x}{l_0 + \xi}, \qquad \text{(d)}$$

$$(l_0 + \xi)^2 = b^2 + x^2. \qquad \text{(e)}$$

Knowing G and c and using the above five equations we can determine five unknowns, i.e. S, N, α, ξ, x. To take into account the friction effects would not complicate the solution at all. Only the Eq. (b) would be changed to $-S\sin\alpha - Nf + G = 0$.

2) PVW solution

In this case, the condition of zero virtual work of active forces is

$$\delta W = G\,\delta x - S\,\delta\xi = 0 .$$

Why does the minus sign appear at the term denoting the virtual work of the spring force? This is due to the fact that the spring force acts against the positive virtual displacement $\partial\xi$ which increases with the increase of δx. Of course, the normal force N does not work since it is perpendicular to the motion of the sleeve.

Since we deal with a mechanism with one degree of freedom, whose position is described by a single parameter, say x, we have to start by finding how the variable ξ depends on x.

From $(l_0 + \xi)^2 = b^2 + x^2$ we get $\xi = -l_0 + \sqrt{b^2 + x^2}$.

The derivative of the previous relation with respect to x and the corresponding variations are

$$\frac{\delta\xi}{\delta x} = \frac{2x}{2\sqrt{b^2 + x^2}} = \frac{x}{\sqrt{b^2 + x^2}} \quad \Rightarrow \quad \delta\xi = \frac{x}{\sqrt{b^2 + x^2}}\delta x .$$

Substituting the constitutive relation $S = c\xi$ into $G\,\delta x - S\,\delta\xi = 0$ we subsequently get

$$G\,\delta x - c\xi\,\delta\xi = 0 ,$$

$$G\,\delta x - c\xi\frac{x}{\sqrt{b^2 + x^2}}\delta x = 0 ,$$

$$G\,\delta x - c\left(-l_0 + \sqrt{b^2 + x^2}\right)\frac{x}{\sqrt{b^2 + x^2}}\delta x = 0 ,$$

$$\left(G - c\left(-l_0 + \sqrt{b^2 + x^2}\right)\frac{x}{\sqrt{b^2 + x^2}}\right)\delta x = 0 .$$

The above relation has to be valid for any virtual displacement δx. From it follows that the outer bracket of the previous relation has to be equal to zero, so

$$G = c\left(-l_0 + \sqrt{b^2 + x^2}\right)\frac{x}{\sqrt{b^2 + x^2}} .$$

We wanted to find how the displacement x depends on the sleeve weight G. Instead, we obtained the relation $G = f(x)$. This function could be readily evaluated by Matlab for varying values of x. The inverse function $x = g(G)$, we were actually looking for, is then obtained graphically. This way, we circumvent rather cumbersome extraction of x form the

resulting formula above. To take into account the friction effects requires solving the task by the FBD procedure first. This would, however, completely disqualify the simplicity of PVW procedure. See the program `S10_spring_sleeve.m.`

```
% S10_spring_sleeve
% original file name is mtl_002_pruzina_objimka
clear
b = 1; c = 1000; l0 = 0.8; G = 100;

x_range = 0:0.1:1;
ksi = -l0 + sqrt(b^2 + x_range.^2)

nom = x_range;
denom = sqrt(b^2 + x_range.^2);

dksi_to_dx = nom./denom;

figure(1)
subplot(2,2,1); plot(x_range, ksi, 'linewidth', 2)
title('ksi as a function of x')
xlabel('x [m]');  ylabel('[m]')
subplot(2,2,2); plot(x_range, dksi_to_dx, 'linewidth', 2)
title('dksi to dx as a function of x')
xlabel('x [m]'); ylabel('[1]')

i = 0;
for x = x_range
    i = i + 1;
    G(i) = c*x*(-l0 + sqrt(b^2  + x^2))/sqrt(b^2 + x^2);
end

subplot(2,2,3); plot(x_range, G, 'linewidth', 2)
title('G as a function of x')
xlabel('x [m]'); ylabel('[N]')

subplot(2,2,4); plot(G, x_range, 'linewidth', 2)
title('x as a function of G')
xlabel('G [N]'); ylabel('[m]')

print -r300 -djpeg mtl_002_pruzina_objimka
```

Graphical output is in Fig S67.

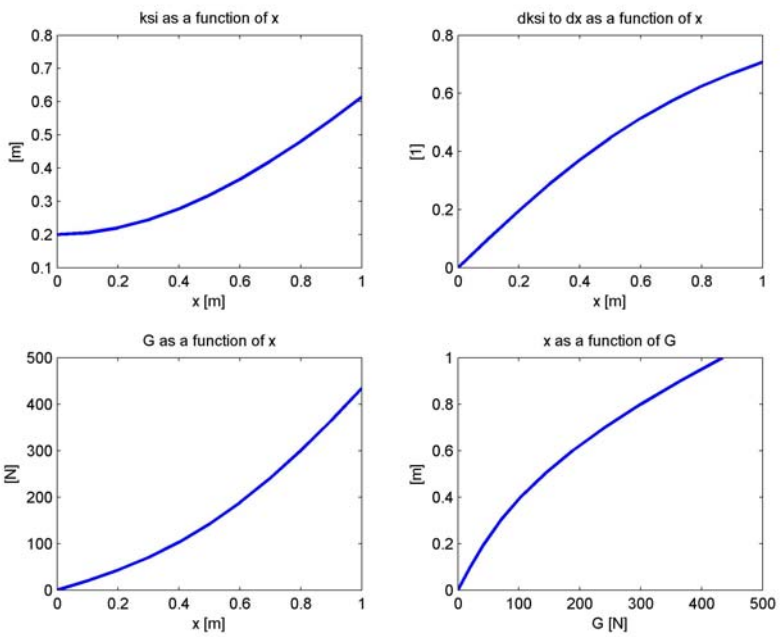


Fig. S67. Matlab output

Example – using PVW find the equilibrium position of a 1 dof system depicted in Fig. S68

Type of task: 1dof system.
Given: Dimensions, spring stiffness k.
Initial length of spring is l_0. Force Q.
Determine: Using PVW, find the equilibrium position.

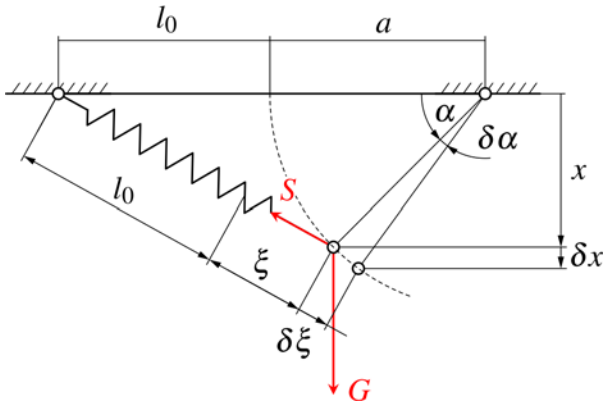


Fig. S68. Principle of virtual work

The condition of zero virtual work due to virtual displacements of the mechanism is

$$\delta W = Q\delta x - S\delta\xi = 0 . \qquad \text{(a)}$$

Note: The axial force in the rotating rod 'does not work' since it is always perpendicular to the trajectory of its end joint where the massless spring is attached. The spring is linear so the spring force $S = k\xi$.

The system has one dof. So, the positional coordinates ξ and the angle α can be uniquely expressed as functions of a single coordinate, say x.

For this, we use the law of cosines

$$l_0 + \xi = \sqrt{(l_0 + a)^2 + a^2 - 2(l_0 + a)a\cos\alpha} . \qquad \text{(b)}$$

The angle α is a function of x as well. It is obvious that

$$\sin\alpha = \frac{x}{a} \quad \Rightarrow \quad \cos\alpha = \pm\sqrt{1 - \left(\frac{x}{a}\right)^2} .$$

Only the plus sign is valid in this case, so

$$\xi = -l_0 + \sqrt{(l_0 + a)^2 + a^2 - 2(l_0 + a)a\sqrt{1 - \left(\frac{x}{a}\right)^2}} = f_1(x) . \qquad \text{(c)}$$

The notation $f_1(x)$ is used for further development.

The derivative of (3) with respect to x is

$$\frac{\delta\xi}{\delta x}=\frac{-2(l_0+a)a\left(\dfrac{-2\dfrac{x}{a}\dfrac{1}{a}}{2\sqrt{1-\left(\dfrac{x}{a}\right)^2}}\right)}{2\sqrt{(l_0+a)^2+a^2-2(l_0+a)a\sqrt{1-\left(\dfrac{x}{a}\right)^2}}}=\frac{(l_0+a)\dfrac{\dfrac{x}{a}}{\sqrt{1-\left(\dfrac{x}{a}\right)^2}}}{\sqrt{(l_0+a)^2+a^2-2(l_0+a)a\sqrt{1-\left(\dfrac{x}{a}\right)^2}}}=f_2(x) \qquad \ldots \text{(d)}$$

The notation $f_2(x)$ is used for further development. Substituting (c) and (d) into (a) and exploiting the spring linearity $S=k\xi$ we get

$$Q=S\frac{\delta\xi}{\delta x}=k\xi\frac{\delta\xi}{\delta x}=k\,f_1(x)\,f_2(x)\,. \qquad \text{(e)}$$

We can evaluate $Q=f(x)$ for varying values of x variable. The Matlab program `S11_principle_of_virtual_work.m` produces Fig. S69.

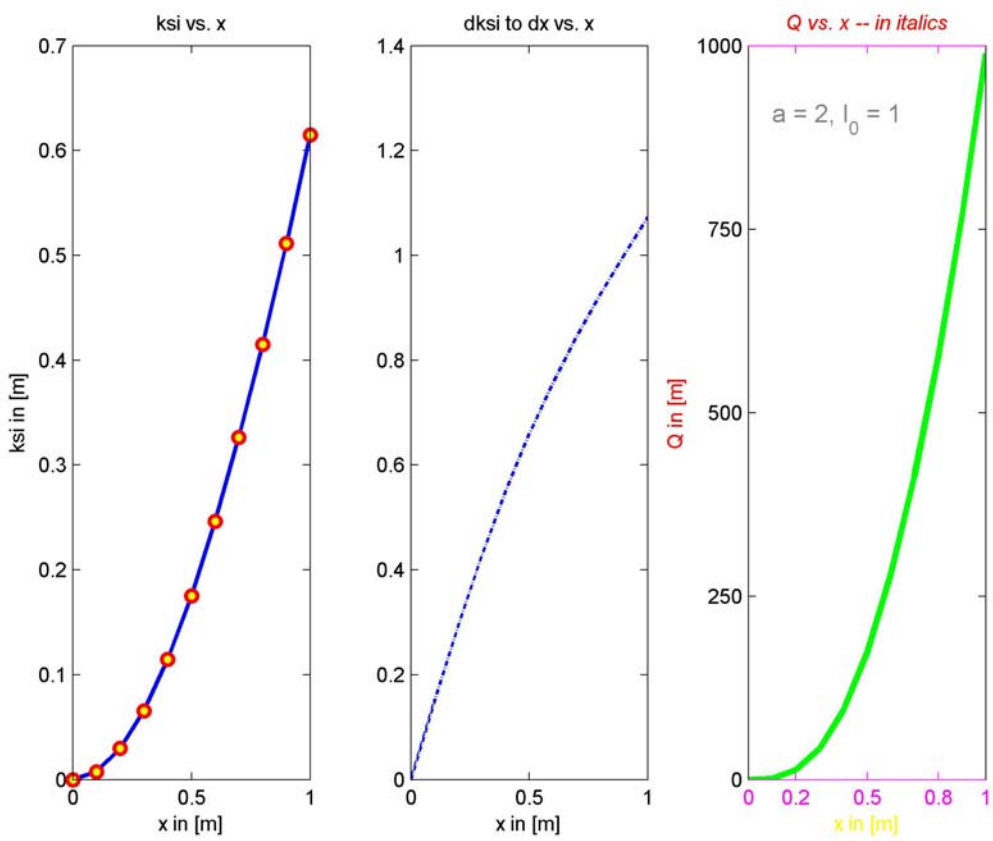


Fig. S69. Matlab output

This Matlab program shows a few nonstandard graphical tricks as well. Just for fun and future convenience.

```
% S11_principle_of_virtual_work
% original_file name is St_princip_virtualnich_praci_priklad_P11
% program requires functions f1 and f2
clear
l0 = 1; a = 2; k = 1500;        % dimensions and spring stiffness
x = 0:0.1:1;
ksi = f1(x,l0,a);               % my function f1
dksi_to_dx = f2(x,l0,a);        % my function f2
figure(1)
subplot(1,3,1)
% markersize should be in multiples of 6
h1 = plot(x,ksi,'-o', 'linewidth', 2, 'markersize',6, ...
    'markeredgecolor', 'r', 'markerfacecolor', 'y')
title('ksi vs. x')
xlabel('x in [m]');
ylabel('ksi in [m]')
subplot(1,3,2)
h2 = plot(x,dksi_to_dx,'-.', 'linewidth', 1.5)
title('dksi to dx vs. x')
xlabel('x in [m]');
Q = k*ksi.*dksi_to_dx;
subplot(1,3,3)
h3 = plot(x,Q)
a3 = get(h3);
title('Q vs. x -- in italics', 'color', 'r', 'fontangle', 'italic');
% color and fonttype for title
txt = ['a = ' num2str(a) ', l_0 = ' num2str(l0)];
text(0.1,900,txt, 'color', [0.5 0.5 0.5], 'fontsize', 12);
% color (gray) and fontsize (14) of text
xlabel('x in [m]','Color','y');         % color of xlabels
% xtick distribution, could be non-uniform
set(gca, 'xtick', [0 0.2 0.5 0.8 1]);
set(gca, 'xcolor', 'm');                % magenta for x axis line and ticks
ylabel('Q in [m]','Color','r');         % color of ylabel
% GCA means Get handle to current axis.
set(gca, 'ytick', [0:250:1000]);        % tick distribution
set(h3, 'linewidth', 3, 'color', 'g')   %linewidth and color for plotted curve
print -djpeg -r300 priklad_P11_fig3

function ksi = f1(x,l0,a)
% it belongs to St_princip_virtualnich_praci_priklad_P11.m
% compute the elongation of spring ksi as a function of x
aa = a^2;
l0pa = l0 + a;
xx = (1 - x.^2/aa).^(0.5);
ksi = -l0 + ((l0pa)^2 + aa - 2*(l0pa)*a*xx).^(0.5);

function dksi_to_dx = f2(x,l0,a)
% it belongs to St_princip_virtualnich_praci_priklad_P11.m
% compute the derivative of ksi with respect to x
aa = a^2;
l0pa = l0 + a;
cit1 = l0pa*x/a;
jm1 = (1 - x.^2/aa).^0.5;
cit = cit1./jm1;
xx = (1 - x.^2/aa).^(0.5);
jm = (l0pa^2 + aa - 2*l0pa*a*xx).^(0.5);
dksi_to_dx = cit./jm;
```

S12. Internal forces

In rigid body mechanics, we are trying to find the state of equilibrium of external and reaction forces acting on bodies regardless of their strength, reliability, durability, etc. The considered bodies are by definition perfectly rigid, that is infinitely stiff, they the do not deform due to the applied forces, and are theoretically indestructible. So, one might naively deduce that what happens inside bodies is generally out of our interest. This would, however, be a shortsighted

approach from the engineering point of view. We know that in practice the bodies break due to applied forces and this evidently happens due to a failure of the weakest part of the internal structure of bodies. What the loading bodies can safely withstand and under what conditions they break is the crucial part of engineering reasoning and is fully treated in mechanics of deformable bodies – that is in chapters devoted to the strength of material.

In statics of rigid bodies, we are capable to determine the internal forces in a chosen part of a body. Here we explain the procedure how to do it. The presented procedure, based on the principles of rigid body mechanics, will become a fundamental step for answering the structural strength tasks in the subject of the strength of material.

The procedure for finding internal forces can be explained studying a simply supported beam of the length l loaded by single force F as depicted in Fig. S70.

Generally, we proceed in three steps.

1. Using FBD we plot the applied and reaction forces.
2. Writing equilibrium conditions and solving them we determine reaction forces.
3. We mentally cut the body in the place of interest, then apply the FBD technique again and determine the internal forces in the cut area by expressing the equivalence of internal forces with those imagined on a chosen side of the cut. Both left- and right-hand part of the body could be alternatively used – both approaches lead to same results.

In this particular case, there are three force effects satisfying the equilibrium of individual parts of the beam. Force T represents the *tensional force* in the cross sectional section of the beam. This force tries to tear the beam at that place apart.

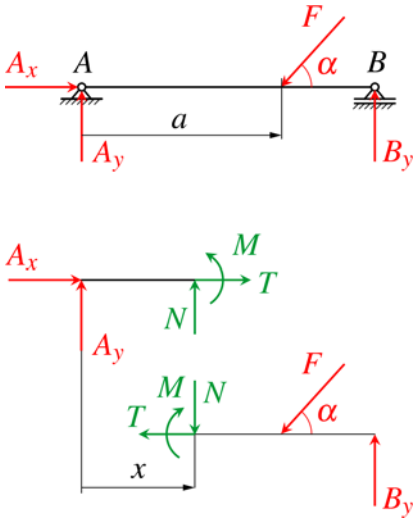


Fig. S70. Internal forces at a cross section

Force N is normal to the beam axis and represents the *shear force*. Finally, the moment M tries to bend the beam – it is called a *bending moment*.

The magnitudes of internal forces acting on divided parts are the same; their directions – in agreement with the principle of action and reaction – are, however, opposite. Of course, the internal forces cancel out when both artificially divided parts are put together. From outside, these forces are not visible – they are internal. Still, to satisfy the engineering requirements concerning the strength of material an observer has to 'immerge' inside the body to find out where are the structural limits of a body to withstand the applied loading. The subject will be treated in more detail in chapters devoted to the strength of material.

Here is the procedure for finding the internal forces in detail.

1. FBD: Applied and reaction forces (in agreement with the fixed joint constraint on the left and the sliding joint constraint on the right) are indicated in the upper part of Fig. S70.

2. The equations of equilibrium written for the whole body are

$$
\begin{aligned}
x: &\quad A_x - F\cos\alpha = 0,\\
y: &\quad A_y - F\sin\alpha + B_y = 0,\\
M_{\mathrm{A}}: &\quad Fa\sin\alpha - B_y l = 0.
\end{aligned}
$$

Solving the system of equations for unknowns we get

$$A_x = F\cos\alpha, \quad A_y = F\left(1+\frac{a}{l}\right)\sin\alpha, \quad B_y = F\frac{a}{l}\sin\alpha.$$

3a. Equivalence of internal forces to known forces on the *left-hand side of the beam*

$$
\begin{aligned}
T &= A_x = F\cos\alpha,\\
N &= A_y = F\left(1+\frac{a}{l}\right)\sin\alpha,\\
M &= -A_y x = -Fx\left(1+\frac{a}{l}\right)\sin\alpha.
\end{aligned}
$$

Notice that the moment of forces about the 'cut' point is evaluated. No other point can be chosen for this purpose.

3b. Equivalence of internal forces to the known forces on the *right hand-side of the beam*

$$
\begin{aligned}
T &= F\cos\alpha,\\
N &= F\sin\alpha - B_y = F\sin\alpha - F\frac{a}{l}\sin\alpha = F\left(1-\frac{a}{l}\right)\sin\alpha,\\
M &= F(a-x)\sin\alpha - B_y(l-x) = F(a-x)\sin\alpha - F\frac{a}{l}(l-x)\sin\alpha = -Fx\left(1+\frac{a}{l}\right)\sin\alpha.
\end{aligned}
$$

Of course, the result has to be same regardless of what the part of the body is analyzed. The magnitudes of internal forces computed with respect to the left and right parts of the body are identical, their directions, indicated by vector arrows, are opposite.

Usually, we treat that part of the body, which requires less menial, algebraic and numerical effort.

Example – internal forces for a simply supported beam

Given: Dimensions, Q_1, Q_2, Q_3, M, x.

Determine: Internal forces of the beam in the I-I cross-section depicted in Fig. S71.

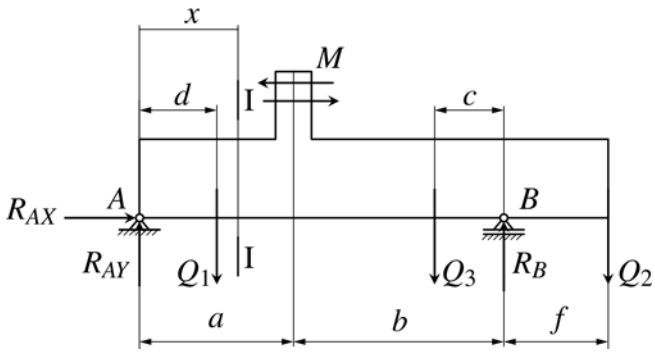


Fig. S71. Simply supported beam

The equilibrium equations for finding reactions are

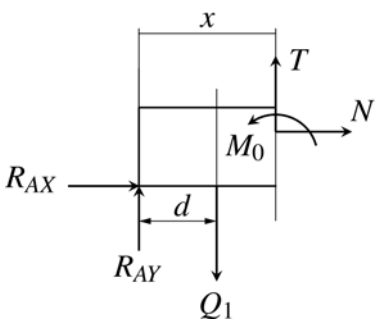
$$x: \quad R_{Ax} = 0,$$

$$y: \quad R_{Ay} - Q_1 - Q_2 + R_{By} - Q_3 = 0,$$

$$M_A: \quad -Q_1 d + M - Q_3(a+b-c) + R_B(a+b) - Q_3(a+b+f) = 0.$$

$$\Rightarrow R_{Ax}, R_{Ay}, R_B.$$

Internal forces at the cross-section I-I, which is located at the distance x from the support A, are obtained by summing up forces and moments along one side of the section. See Fig. S72. First, take the left part of the beam and write the equivalence equations for establishing normal and shear forces N, T and the bending moment M_o.



$$N = R_{Ax},$$

$$T = R_{Ay} - Q_1,$$

$$M_o = -R_{A_y} x + Q_1(x-d).$$

Fig. S72. Internal forces in the I-I cross section

Varying the distance x from zero to $(a+b+f)$, the following diagram for the distribution of the shear force $T(x)$ and the bending moment $M_0(x)$ as functions of the longitudinal variable x can be obtained. See Fig. S73.

This subject will be more closely analyzed in chapters devoted to the mechanics of deformable bodies.

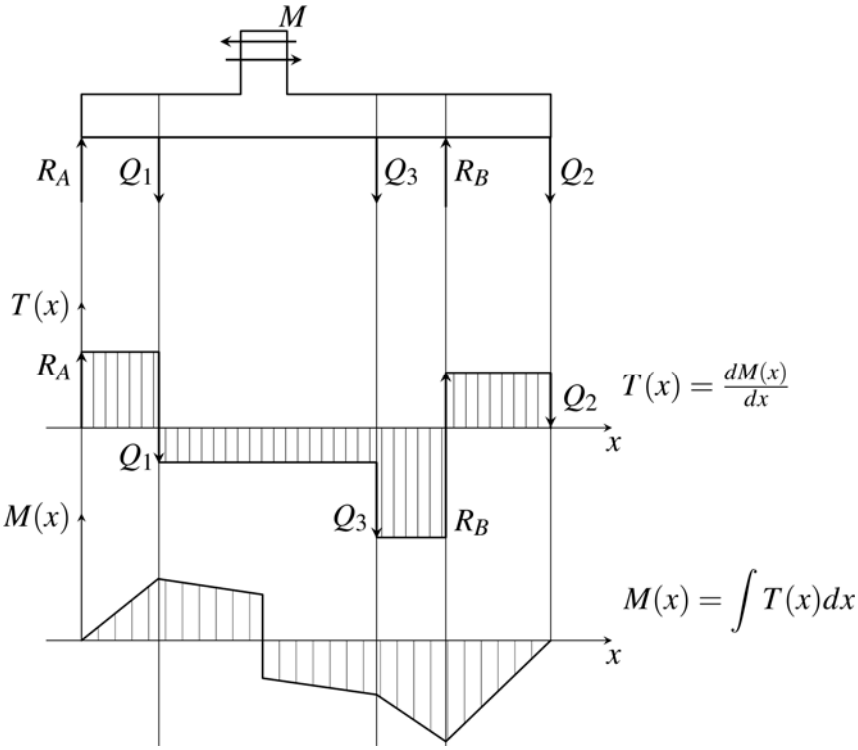


Fig. S73. Shear forces and bending moments along the beam

Example – internal forces along a cantilever beam. See Fig. S74.

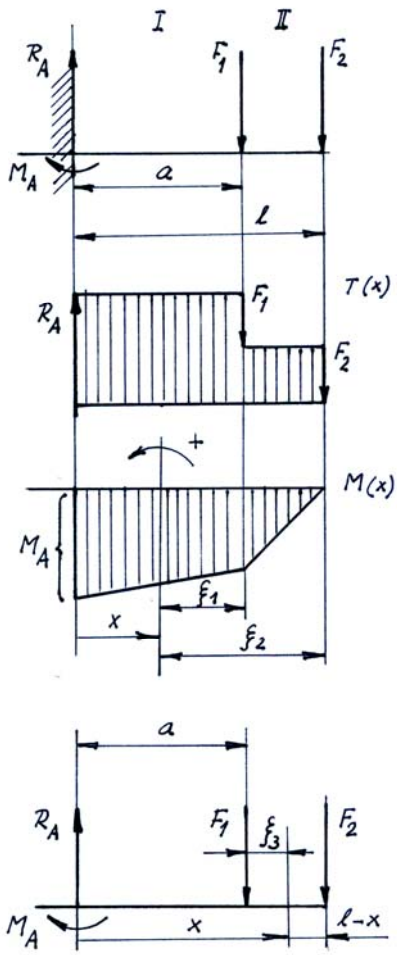
Given: Cantilever beam loaded by two forces F_1, F_2.
Determine: Shear force and bending moment as functions of longitudinal variable x.

The equilibrium conditions based on the FBD are

$0 = 0$ ... there are no forces in this direction,
$R_A - F_1 - F_2 = 0,$
$M_A + F_1 a + F_2 l = 0.$

There are only two reaction forces in this case, i.e.

$R_A = F_1 + F_2 = 0,$
$M_A = -F_1 a - F_2 l.$



Fig, S74. Cantilever beam – internal forces

Area I for $0 \le x \le a$.

Considering right-hand forces – counterclockwise

$$
\begin{aligned}
M_{\mathrm{I}}(x) &= -F_1\xi_1 - F_2\xi_2 = \\
&= -F_1(a-x) - F_2(l-x) = \\
&= \underbrace{-F_1 a - F_2 l}_{M_{\mathrm{A}}} + \underbrace{(F_1+F_2)}_{R_{\mathrm{A}}}x = M_{\mathrm{A}} + R_{\mathrm{A}}x.
\end{aligned}
$$

$$
\begin{aligned}
x + \xi_1 = a \quad &\Rightarrow \quad \xi_1 = a - x, \\
x + \xi_2 = l \quad &\Rightarrow \quad \xi_2 = l - x.
\end{aligned}
$$

Considering left-hand side forces – clockwise

$$M_{\mathrm{I}}(x) = M_{\mathrm{A}} + R_{\mathrm{A}}x.$$

Area II, for $a \le x \le l$, counterclockwise

$$M_{\mathrm{II}}(x) = -F_2(l-x).$$

Clockwise

$$
\begin{aligned}
M_{\mathrm{II}}(x) &= M_{\mathrm{A}} + R_{\mathrm{A}}x - F_1\xi_3 = M_{\mathrm{A}} + R_{\mathrm{A}}x - F_1(x-a) = \underbrace{M_{\mathrm{A}} + F_1 a}_{-F_1 a - F_2 l + F_1 a} + \underbrace{(R_{\mathrm{A}} - F_1)}_{F_2}x = \\
&= -F_2 l + F_2 x = -F_2(l-x) = F_2(x-l).
\end{aligned}
$$

$$\xi_3 + a = x \quad \Rightarrow \xi_3 = x - a.$$

The results have to be same regardless of the considered part being treated. Varying x, we can plot the distribution of the shear force $T(x)$ and the bending moment $M(x)$ as functions of x as shown in Fig. S74.

Example – internal forces due to continuous loading

Given: Simply supported beam, continuous loading – $l, q_1, q_{\max}$. See Fig. S75.
Determine: Reactions and bending moment and shear force as functions of the longitudinal variable x.

By the continuous loading we understand the cumulative weight effect of a homogeneous layer of a loose aggregation of substances as sand, snow, gravel, etc. In Fig. S75 there are indicated two layers; the lower one, which is of constant 'height' and the upper one, which has a triangular shape.

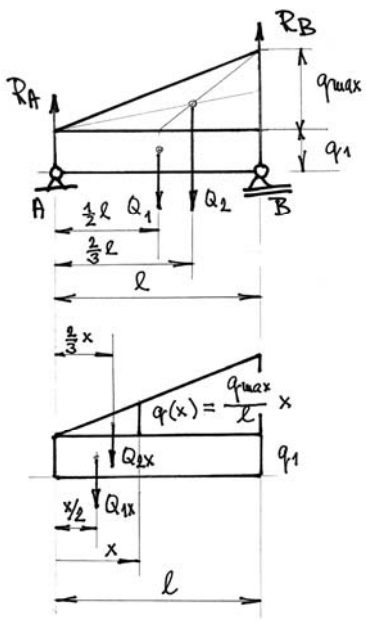


Fig. S75. Internal forces in a continuously loaded beam

In 2D cases the ‘loading density’ is expressed by quantities measured in $[\mathrm{N/m}]$.

For purposes of rigid body mechanics, the effect of continuous loading is replaced by an equivalent resulting force, which acts in the centre of gravity of the graphical representation of the continuous loading. In our case we have two layers to which two equivalent forces, say Q_1, Q_2, are assigned.

Observing FBD reasoning we can write

$$Q_1 = q_1 l\,,$$
$$Q_2 = \frac{1}{2} q_{\max} l\,.$$

First, evaluate the reaction forces. The equations of equilibrium are

$$x: \quad 0 = 0,$$
$$M_{\mathrm{A}}: \quad Q_1 \frac{l}{2} + Q_2 \frac{2}{3} l - R_{\mathrm{B}} l = 0,$$
$$M_{\mathrm{B}}: \quad -Q_1 \frac{l}{2} - Q_2 \frac{1}{3} l + R_{\mathrm{A}} l = 0.$$

Solving them gives the unknown reaction forces

$$R_{\mathrm{A}} = l\left(\frac{q_1}{2} + \frac{q_{\max}}{6}\right), \quad R_B = l\left(\frac{q_1}{2} + \frac{q_{\max}}{3}\right).$$

Now, assume that we intend to express the internal forces as functions of longitudinal coordinate, say x. The partial equivalents of equivalent loading within the interval $\langle 0, x \rangle$ are

$$Q_{1x} = q_1 x,$$
$$Q_{2x} = \frac{1}{2} x q_{\max} \frac{x}{l} = \frac{1}{2} \frac{q_{\max}}{l} x^2.$$

Observing the acting forces and moments along the interval $\langle 0, x \rangle$ and assigning them to the sought-after internal forces we get distributions of the normal force, the shear force and the bending moment as functions of time.

1. Axial force $\quad N(x) = 0\,.$
2. Shear force $\quad T(x) = R_{\mathrm{A}} - q_1 x - \frac{1}{2} q_{\max} \frac{x^2}{l}\,.$
3. Bending moment $\quad M(x) = R_{\mathrm{A}} x - \frac{1}{2} q_1 x^2 - \frac{1}{3} q_{\max} \frac{x^3}{l}\,.$

Note: The magnitudes of $R_{\mathrm{A}}, R_{\mathrm{B}}$ are already known.

S13. Centre of gravity, centre of mass and static moment of area

The *centre of gravity* of a body is a point where the resultant force of gravity (weight) forces of individual body's elements is located. We could also define the centre of gravity as a point around which the resultant moment of gravity (weight) forces of all the individual body's elements is identically equal to zero.

Another point of view. The *centre of gravity* is a point in which the overall effect of the resultant gravity force is the same as the effect of gravitational forces acting on individual body's elements. If the body has to be suspended or supported, at a single point, then the so-called *axis of centre of gravity* has to pass through the centre of gravity and the suspension or supporting point.

The *centre of mass* is a more general term. It is associated only with the body's geometrical shape and with the density distribution. The location of the centre of mass is independent of the surrounding gravitational field. Furthermore, the centre of mass, in contradistinction of the centre of gravity, does not depend on the fact whether the body 'lives' in the inertial or non-inertial system.

In a pseudo inertial frame of reference – in the vicinity of the Earth's surface – where the gravitational field is considered to be uniformly distributed, the centre of mass is practically identical with that of gravity.

In deep space, where no gravity could be assumed, the term centre of gravity loses its meaning.

Realizing the subtle differences, we might use both terms interchangeably. In the text, however, the term centre of mass is preferred.

It should be reminded that matter has different properties – among them weight and mass.

Weight	… $G = mg$	… mass × gravitational acceleration.
Mass	… $m = \rho V = \int \rho \, \mathrm{d}V = \int \rho \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$	… density × volume.
Volume	... $V = Sl$	… for example area × length.

If the density of a body is considered constant, then the location of the centre of mass could be computed using mass, weight or volumetric approach. Also, the so called *static moments* of area or volume might help.

Location of the centre of gravity of a planar object using the static moments of area

Observing Fig. S76 we define

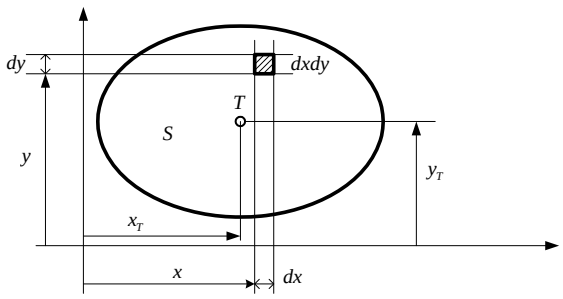


Fig. S76. Centre of mass

$$M = \int_S \mathrm{d}x\,\mathrm{d}y \quad \text{... area,}$$

$$M_x = \int_S y\,\mathrm{d}x\,\mathrm{d}y \text{...static moment of area about } x \text{ axis,}$$

$$M_y = \int_S x\,\mathrm{d}x\,\mathrm{d}y \text{...static moment of area about } y \text{ axis,} \qquad \text{... (S13\_1)}$$

$$y_\mathrm{T} M = M_x \text{... static moment of the whole area} = \text{static moment of its parts,}$$

$$\Rightarrow \text{the } y \text{ coordinate of the centre of gravity is } y_\mathrm{T} = \frac{M_x}{M}.$$

In general, we define the static moments of area about coordinate axes by

$$M_x = \int y\,\mathrm{d}m ... \int y\,\mathrm{d}V ... \int y\,\mathrm{d}S \;\ldots \text{ mass, volumetric and area static moments about } x \text{ axis,}$$

$$M_y = \int x\,\mathrm{d}m ... \int x\,\mathrm{d}V ... \int x\,\mathrm{d}S \;\ldots \text{ mass, volumetric and area static moments about } y \text{ axis.}$$

... (S13_2)

$$m\,x_\mathrm{T} = \int y\,\mathrm{d}m \Rightarrow x_\mathrm{T} = \frac{1}{m}\underbrace{\int y\,\mathrm{d}m}_{M_x} = \frac{1}{m}M_x,\; m\,y_\mathrm{T} = \int x\,\mathrm{d}m \Rightarrow y_\mathrm{T} = \frac{1}{m}\underbrace{\int x\,\mathrm{d}m}_{M_y} = \frac{1}{m}M_y.$$

... (S13_3)

So, the determination of the location of the centre of mass is based on the statement that the static moment of the whole body is equal to the sum of static moment of individual parts. For the computation, we can use mass, weight, volumetric or area elements. If the gravitational field is uniform and the density homogeneous, then all the mentioned approaches lead to the same result.

Example – centre of mass coordinates

Given: Quarter of a circle.
Determine: The coordinates of the centre of mass for a quarter circle depicted in Fig. S77.

The area of a quarter circle is

$$M = \pi R^2/4.$$

The static moment of the quarter circle about the x axis is considered as the integral sum of moments of its infinitesimal areas defined by $dx \times dy$. So,

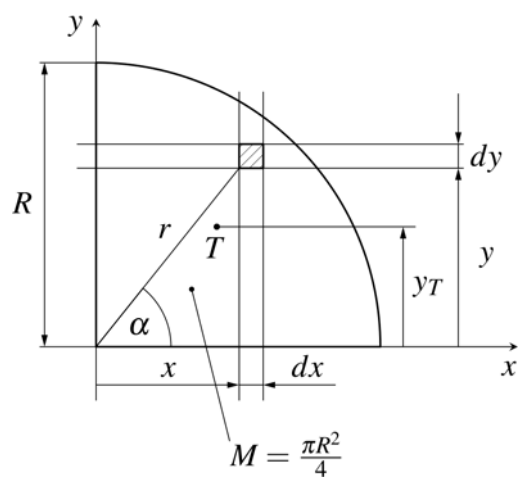


Fig. S77. Quarter circle – centre of mass

$$M_x = \int_{x=0}^{R} \int_{y=0}^{\sqrt{R^2-x^2}} y\,\mathrm{d}x\,\mathrm{d}y.$$

It is convenient to evaluate the integral by means of polar coordinates. The change of variables behind the integral operator requires taking the coordinate transformation and the corresponding Jacobian into account.

So, $x = r\cos\alpha,\ y = r\sin\alpha,\ J = r$ ... Jacobian.

$$M_x = \int_{r=0}^{R}\int_{\alpha=0}^{\pi/2} r^2 \sin\alpha \,\mathrm{d}r\,\mathrm{d}\alpha = \int_{\alpha=0}^{\pi/2} \frac{R^3}{3}\sin\alpha\,\mathrm{d}\alpha = -\frac{R^3}{3}\left[\cos\alpha\right]_0^{\pi/2} = \frac{R^3}{3}.$$

Since the static moment of the whole body is equal to the sum of static moments of individual infinitesimal elements, we can express the y coordinate of the centre of mass from

$$M\,y_\mathrm{T} = M_x \Rightarrow y_T = \frac{M_x}{M} = \frac{R^3/3}{\pi R^2/4} = \frac{4R}{3\pi}.$$

The object is symmetrical so, the x_T coordinate is the same.

Alternatively, the static moment could evaluated from 'elementary slices' as

$$M_x = \int y\,\mathrm{d}x\frac{y}{2} = \frac{1}{2}\int y^2\mathrm{d}x \text{ and since } y^2 = R^2 - x^2 \text{ we can rewrite it into}$$

$$M_x = \frac{1}{2}\int_0^R \left(R^2 - x^2\right)\mathrm{d}x = \frac{1}{2}\left[R^2 x - \frac{x^3}{3}\right]_0^R = \frac{1}{2}\frac{2}{3}R^3 = \frac{R^3}{3}.$$

Example – centre of mass coordinates of a blade

Given: The left boundary of the blade is formed by a line defined by $y = k_1 x$, the right boundary is a parabola defined by $y = k_2 x^2$. The upper boundary is formed by a constant line $y = b$. For given values k_1, k_2 and for the prescribed value b we get reference dimensions $a = b/k_1$ and $c = \sqrt{b/k_2}$, respectively. See Fig. 78. The input values have to be carefully chosen in such a way that the b value is 'below' the intersection of both curves. See Fig. 79.
Determine: The coordinates of the centre of mass.

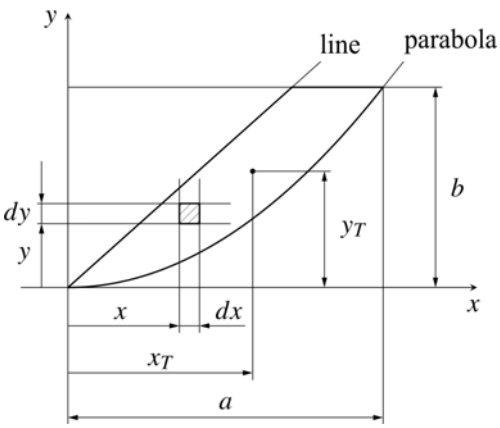


Fig. S78. Blade – centre of mass

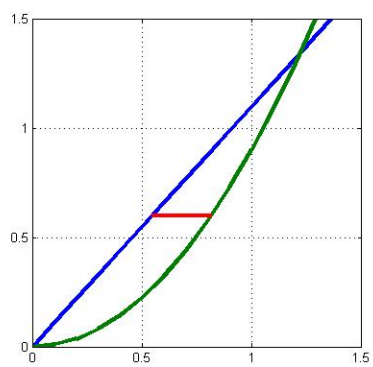


Fig. 79. Solution limits

In this case, the blade area the can be evaluated as

$$M=\int_0^a\int_{bx^2/a^2}^{bx/c}\mathrm{d}x\,\mathrm{d}y=\int_0^a\left[y\right]_{bx^2/a^2}^{bx/c}\mathrm{d}x=\int_0^a\left(\frac{bx}{c}-\frac{bx^2}{a^2}\right)\mathrm{d}x=\left[\frac{bx^2}{2c}-\frac{bx^3}{3a^2}\right]_0^a=\frac{ba^2}{2c}-\frac{ba^3}{3a^2}.$$

And the static moment about the x axis is

$$M_x=\int_0^a\int_{bx^2/a^2}^{bx/c}y\,\mathrm{d}x\,\mathrm{d}y=\int_0^a\left[\frac{y^2}{2}\right]_{bx^2/a^2}^{bx/c}\mathrm{d}x=\frac{1}{2}\int_0^a\left(\frac{b^2x^2}{c^2}-\frac{b^2x^4}{a^4}\right)\mathrm{d}x=\frac{1}{2}\left[\frac{b^2x^3}{3c^2}-\frac{b^2x^5}{5a^4}\right]_0^a=$$

$=\frac{1}{2}\left[\frac{b^2a^3}{3c^2}-\frac{b^2a^5}{5a^4}\right]$. So, the y coordinate of the centre of gravity is $y_\mathrm{T}=\frac{M_x}{M}$.

Example – volume of sphere

Given: A sphere is depicted in Fig. S80.
Determine: The volume of a sphere. Everybody knows the answer by heart, but do it anyway in order to recal how the transformation into spherical coordinates is provided.

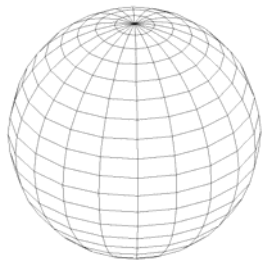


Fig. S80. Sphere

The volume of a body R with properly defined limits is

$$V=\iiint\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,.$$

Often, it is convenient to solve the tasks using the transformation of coordinates into the polar coordinate system. See Fig. S81. In this case the transformation has the form

$$x=r\sin\vartheta\cos\alpha,$$
$$y=r\cos\alpha,$$
$$z=r\sin\vartheta\sin\alpha.$$

When the change of variables behind the integral sign is carried out, the Jacobian of the transformation has to be added. In this case $J=r^2\sin\vartheta$.

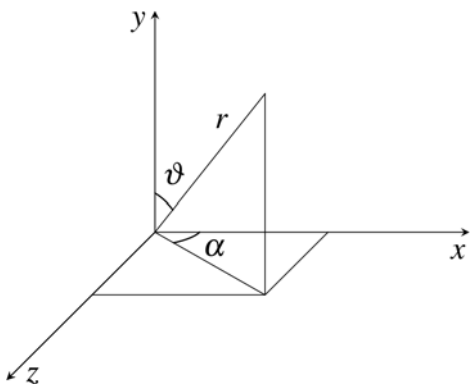


Fig. S81. Spherical coordinates

To simplify the computation we could evaluate the volume of a sphere as the eight-multiple of the volume of one eighth of the sphere in the first quadrant. See Fig. S82.

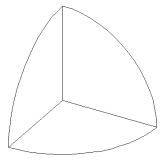


Fig. S82. One eighth

$$V = \iiint \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 8\int_{r=0}^{R}\int_{\alpha=0}^{\pi/2}\int_{\vartheta=0}^{\pi/2} r^2 \sin\vartheta\,\mathrm{d}r\,\mathrm{d}\alpha\,\mathrm{d}\vartheta = 8\int_0^R r^2\mathrm{d}r \int_0^{\pi/2}\mathrm{d}\alpha \int_0^{\pi/2}\sin\vartheta\,\mathrm{d}\vartheta = 8\frac{R^3}{3}\frac{\pi}{2}\left[-\cos\vartheta\right]_0^{\pi/2} = \frac{4}{3}\pi R^3$$

Example – just for fun – plot one eighth of sphere in Matlab. See Fig. S83.

```
% one_eighth
clear
[X,Y,Z] = sphere;
figure(1)
surf(X,Y,Z); axis equal; axis off

szX = size(X);
for i = 1:21
    for j = 1:21
        if X(i,j) < 0, X(i,j) = 0; end
 %      if Y(i,j) < 0, Y(i,j) = 0; end
        if Z(i,j) < 0, Z(i,j) = 0; end
    end
end

figure(2)
surf(X,Y,Z); axis square; axis equal; xlabel('x'); ylabel('y')

figure(3)
x = [0 1]; y = [0 0]; z = [0 0];
plot3(x,y,z,'k'); axis square; axis equal;
xlabel('x'); ylabel('y'); grid; zlabel('z'); axis off
az = 120; el = 30; view(az,el)
hold on

x1 = [0 0]; y1 = [0 1]; z1 = [0 0];
plot3(x1,y1,z1,'k'); hold on

phi = 0:pi/32:pi/2; d = length(phi); r = 1;
x2 = r*sin(phi); y2 = r*cos(phi); z2 = r*zeros(1,d);
plot3(x2,y2,z2,'k'); hold on

plot3(z2,x2,y2,'k'); hold on

plot3(y2,z2,x2,'k'); hold on

x3 = [0 0]; y3 = [0 0]; z3 = [0 1];
plot3(x3,y3,z3,'k'); hold off
```

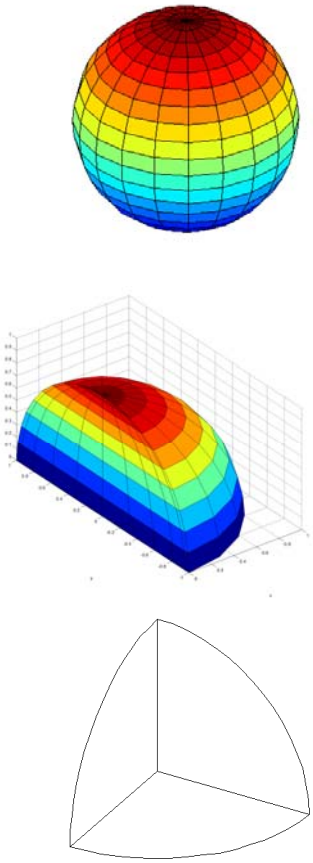


Fig. S83.

14. References

[1] *Encyclopedia of Physics*, Vol. III/1, Editted by Fluege, Springer, Berlin, 1960.

[2] Mase, G.R.: *Theory and Problems of Continuum mechanics*, Schaum's Outline Series, Mc Graw-Hill, 1970.

[3] Book 11 of St. Augustine's Confessions.
See also: www9.georgetown.edu/faculty/jod/latinconf/11.html

[4] Cornelius Lancosz: *The variational principles of mechanics*, University of Toronto Press, 1949.
See also: https://archive.org/details/VariationalPrinciplesOfMechanicsLanczos

[5] R. Pratap and A. Ruina: *Introduction to statics and dynamics*, Oxford University Press, 2001.
See also: www.docdatabase.net/more-introduction-to-statics-dynamics-chapters-1-10-1242197.html.

[6] R. Feynman: *Lectures on physics*, http://www.feynmanlectures.caltech.edu

[7] Stejskal, V., Dehombreux P., Eiber, A., Gupta, R., Okrouhlik, M.: *Mechanics with Matlab*, Electronic Textbook , ISBN 2-9600226-2-9, http://www.geniemeca.fpms.ac.be, Faculté Polytechnique de Mons, Belgium, April 2001.

[8] Okrouhlik, M. (editor): *Numerical methods in computational mechanics*. Institute of Thermomechanics, Prague 2009, ISBN 978-80-87012-35-2.
See also http://www.it.cas.cz/files/u1784/Num_methods_in_CM.pdf

[9] Okrouhlik, M.: *Implementation of Nonlinear Continuum Mechanics in Finite Element Codes*. Publications of the Institute of Thermomechanics, Educational Series. ISBN 80-85918-06-4, 256 p. Prague, 1995.
See also: IMPLEMENTATION_of_NONLINEAR_Cont_Mech_1995.pdf

[10] Christoph Schiller: *Motion, Mountain – the adventure of physics*.
See also: www.motionmountain.net/author.html

Kinematics

Scope

1. Introduction to kinematics
2. Motion of particles
3. Rotary and translatory motion of bodies
4. Acceleration components appearing in a non-inertial frame of reference
5. Generic motion of bodies in two-dimensional space
6. References

K1. Introduction to kinematics

Kinematics is a subject of classical mechanics which deals with quantities describing the motions of particles and bodies, without considering the causes that induce the motion. These quantities, i.e. displacement, velocity (time rate of displacement) and acceleration (time rate of velocity), are measured in $[\mathrm{m}]$, $[\mathrm{m/s}]$ and in $[\mathrm{m/s^2}]$ respectively. Kinematics tools, together with those of statics, are necessary instruments for solving problems of dynamics. A reader is recommended to enlarge his views studying the textbooks listed in References.

K2. Motion of particles

K2.1. Motion along a straight line

In this case, a single spatial variable, say x, suffices for a unique determination of the particle position. We say that this case has one degree of freedom.

Knowing the location or the displacement measured from a certain origin, as a function of time

$$x = x(t)\,, \tag{K\_1}$$

then we define the immediate, or the current velocity as a time rate of displacement, or by other words, as the first derivative of displacement with respect time

$$v = \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}\,. \tag{K\_2}$$

The instantaneous velocity should be distinguished from the average velocity, say v_{avg}, which is obtained as a sum of the velocity v_{E} measured at the time t_{E}, plus the velocity v_{B} measured at the time t_{B}, and divided by the corresponding time interval $\Delta t = t_{\mathrm{E}} - t_{\mathrm{B}}$. So,

$$v_{\mathrm{avg}} = \frac{v_{\mathrm{E}} + v_{\mathrm{B}}}{\Delta t}\,. \tag{K\_3}$$

Generally, when the term velocity is used, it is understood that the immediate velocity is meant.

Acceleration (again, the immediate acceleration or the acceleration right now) is defined as the first derivative of velocity with respect to time or as the second derivative of displacement with respect to time.

$$a = \dot{v} = \frac{\mathrm{d}v}{\mathrm{d}t}, \qquad \text{(K\_4)}$$

$$a = \ddot{x} = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}. \qquad \text{(K\_5)}$$

Eliminating time variable from (K_4) and (K_5) we get

$$a = \frac{v\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}v^2}{2\mathrm{d}x}. \qquad \text{(K\_6)}$$

The motion of a particle moving along a straight line might be classified as follows

K2.1.1. Motion with constant velocity – $v = const$.

If $v = const = c$ then $\frac{\mathrm{d}v}{\mathrm{d}t} = a = 0$.

For initial conditions, $t = t_0$, $x = x_0$, we get $x = x_0 + c(t - t_0)$. (K_7)

K2.1.2. Motion with a constant acceleration – $a = const$.

Let $a = const = k$; then for initial conditions $t = t_0$, $x = x_0$, $v = v_0$, we get

$v = v_0 + k(t - t_0)$, for zero initial conditions we get $v = at$, (K_8a)

$x = x_0 + v_0(t - t_0) + \frac{1}{2}k(t - t_0)^2$, for zero initial conditions we get $x = \frac{1}{2}at^2$. (K_8b)

Example – uniformly accelerating motion

Given: A motion of a particle with a constant acceleration along the straight line is assumed.
Determine: Derive formulas for velocity and acceleration. For given initial conditions plot the distributions of displacement, velocity and acceleration as functions of time.

The velocity distribution is obtained by integrating the relation $\frac{\mathrm{d}v}{\mathrm{d}t} = a = const$.

$$\int_{v_0}^{v} \mathrm{d}v = \int_{t_0}^{t} a\,\mathrm{d}t,$$

$$v - v_0 = a(t - t_0),$$

$$v = v_0 + a(t - t_0). \qquad \text{(K\_9)}$$

The displacement distribution is obtained by integrating the relation $\frac{\mathrm{d}s}{\mathrm{d}t} = v$.

$$\int_{s_0}^{s} \mathrm{d}s = \int_{t_0}^{t} v_0 + a(t-t_0)\,\mathrm{d}t\,,$$

$$s = s_0 + v_0(t-t_0) + \frac{1}{2}a(t-t_0)^2\,. \qquad \text{(K\_10)}$$

Defining initial conditions by

```
t0 = 1; t = t0:0.1:2; a = 5; v0 = 3; s0 = 1;
```

we can write

```
v = v0 + a*(t-t0);
s = s0 + v .* (t-t0) + 0.5*a*(t-t0).^2;
```

Eliminating time variable from Eqs. (K_9) and (K_10) we get the velocity as a function of displacement

$$v^2 - v_0^2 = 2a(s-s_0)\,.$$

The same result can be alternatively obtained by integrating the relation

$$\mathrm{d}v^2 = 2a\,\mathrm{d}s\,.$$

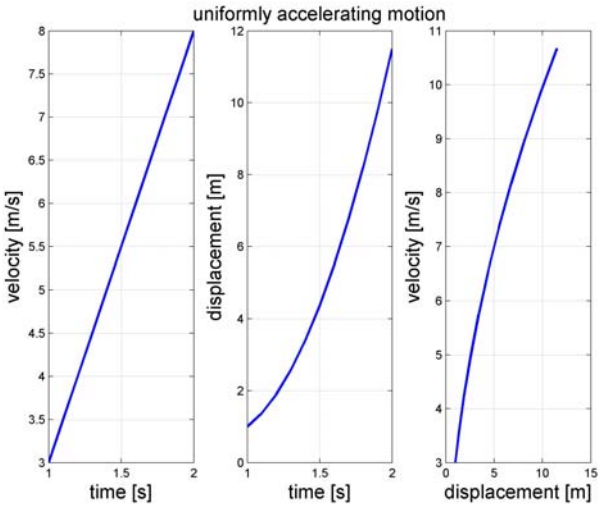


Fig. K01. Displacements and velocities for uniform acceleration

See the program `K01_uniformly_accelerating_motion` and its graphical output Fig. K01.

```
% K01_uniformly_accelerating_motion
% old file name is m_007_rovnomerne zrychleny_pohyb po primce_en.m
% constant acceleration a

clear
t0 = 1; t = t0:0.1:2;        % time range
a = 5; v0 = 3; s0 = 1;       % given acceleration and initial conditions
v = v0 + a*(t-t0);           % velocity as a function of time
s = s0 + v .* (t-t0) + 0.5*a*(t-t0).^2; % displacement as a function of time

v1 = sqrt(v0^2 + 2*a*(s-s0));   % velocity as a function of displacement

figure(1)
subplot(1,3,1); plot(t,v, 'linewidth',2);
ylabel('velocity [m/s]', 'fontsize', 16);
xlabel('time [s]', 'fontsize', 16); grid
subplot(1,3,2); plot(t,s, 'linewidth',2);
title('uniformly accelerating motion', 'fontsize', 16)
ylabel('displacement [m]', 'fontsize', 16);
xlabel('time [s]', 'fontsize', 16); grid;
subplot(1,3,3); plot(s,v1, 'linewidth',2);
ylabel('velocity [m/s]', 'fontsize', 16); grid;
xlabel('displacement [m]', 'fontsize', 16);

print -djpeg -r300 f_007_1_en

% end of m_007_rovnomerne zrychleny_pohyb po primce_en.m
```

Example – motion with cubic increase of displacement

Given: $x(t) = x_0 + kt^3$, where k, x_0 are constants.
Determine: $v(t), a(t), v(a), s(a), s(v)$.

$$x = x(t) = x_0 + kt^3, \tag{a}$$
$$v = v(t) = 3kt^2, \tag{b}$$
$$a = a(t) = 6kt. \tag{c}$$

The functional dependence of individual quantities on time is not always explicitly stated. Often, we simplify the notation by writing $x = x(t)$, etc.

Eliminating the time variable from the last equation, i.e. $t = \frac{a}{6k}$, and substituting it into the last but one equation we get the formula for the velocity as a function of time in the form
$v = \frac{a^2}{12k}$.

Similarly, we could obtain the displacement as a function of acceleration
$x = x_0 + k\frac{a^3}{6^3k^3} = x_0 + \frac{a^3}{216k^2}$.

And finally, eliminating the time variable from Eq. (b) $t = \sqrt{\frac{v}{3k}}$ and substituting it into (a) we get
$x = x_0 + k\sqrt{\frac{v^3}{27k^3}} = x_0 + \sqrt{\frac{v^3}{27k}}$.

Example – the motion with decreasing velocity

Given: $a = -a_0 - kv$, $a_0, k = const$, initial velocity v_0.
Determine: The distance x_s, where the current velocity reaches just the half of initial velocity, i.e. $v_0/2$.

$$a = -a_0 - kv, \tag{a}$$
$$\frac{v\mathrm{d}v}{\mathrm{d}x} = -a_0 - kv, \tag{b}$$
$$-\int_{v_0}^{v_0/2} \frac{v}{a_0 + kv}\mathrm{d}v = \int_0^{x_s} \mathrm{d}x. \tag{c}$$

It should be reminded that

$$\int \frac{\mathrm{d}x}{x} = \lg(x),$$
$$\int \frac{\mathrm{d}x}{kx} = \frac{1}{k}\lg(x) \qquad \text{... multiplication constant,}$$

$$\int \frac{dx}{a+x} = \lg(a+x) \qquad \ldots \qquad \text{substitution } z = a+x,\ \mathrm{d}z = \mathrm{d}x\,,$$

$$\int \frac{\mathrm{d}x}{a+bx} = \frac{1}{b}\lg(a+x) \qquad \ldots \qquad \text{substitution } z = a+bx,\ b\mathrm{d}x = \mathrm{d}z\,.$$

The integral $\int \lg(x)\,\mathrm{d}x$ can be evaluated by the 'per partes' rule[1] according to $\int u'v = uv - \int uv'$. In our case,

$$u' = 1,\ v = \lg(x),$$
$$u = x,\ v' = \frac{1}{x}.$$

and so,

$$\boxed{\int \lg(x)\,\mathrm{d}x = x\lg(x) - \int x\frac{1}{x}\mathrm{d}x = x\lg(x) - x.}$$

Check. Knowing that the derivative of a product is $(uv)' = u'v + uv'$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(x\lg(x) - x) = 1\times\lg(x) + x\frac{1}{x} - 1 = \lg(x)\,.$$

Similarly,

$$\boxed{\begin{aligned}&\int \lg(a+bx)\mathrm{d}x\cdots\big(\text{po substituci } (a+bx = z,\ b\mathrm{d}x = \mathrm{d}z,\ \mathrm{d}x = \mathrm{d}z/b\big)\cdots = \frac{1}{b}z\big[\lg(z) - 1\big] = \\ &= \frac{1}{b}(a+bx)\big[\lg(a+bx) - 1\big] = \frac{a}{b}\lg(a+bx) - \frac{a}{b} + x\lg(a+bx) - x.\end{aligned}}$$

Another case

$\int \frac{x}{a+bx}\mathrm{d}x$ by per partes rule $\int u'v = uv - \int uv'$ gives

$$u' = \frac{1}{a+bx}, \qquad v = x,$$
$$u = \frac{1}{b}\lg(a+bx), \qquad v' = 1.$$

[1] Also called integration by parts

$$\int \frac{x}{a+bx}\mathrm{d}x = \frac{x}{b}\lg(a+bx) - \int \frac{1}{b}\lg(a+bx)\mathrm{d}x =$$

$$\frac{x}{b}\lg(a+bx) - \frac{1}{b}\left[\left(\frac{a}{b}+x\right)\lg(a+bx) - \frac{a}{b} - x\right] =$$

$$\frac{x}{b}\lg(a+bx) - \left[\left(\frac{a}{b^2}+\frac{x}{b}\right)\lg(a+bx) - \frac{a}{b^2} - \frac{x}{b}\right] =$$

$$-\frac{a}{b^2}\lg(a+bx) + \frac{a}{b^2} + \frac{x}{b} = \frac{a}{b^2}\left(1-\lg(a+bx)\right) + \frac{x}{b}.$$

Check. The derivative of the result gives the initial term

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{a}{b^2}\left(1-\lg(a+bx)\right)+\frac{x}{b}\right) = -\frac{a}{b^2}\frac{b}{a+bx} + \frac{1}{b} = -\frac{a}{b}\frac{1}{a+bx} + \frac{1}{b} = \frac{-a+a+bx}{b(a+bx)} = \frac{x}{a+bx}.$$

Matlab provides the result differing by an integration constant only

```
y1 = x/(a+b*x)
int_y1 = 1/b*x-a/b^2*log(a+b*x)

>> pretty(int_y1)
```

```
                    a log(a + b x)
x/b - -------------- ...
                  2
                 b
```

$$= \frac{x}{b} - \frac{a}{b^2}\lg(a+bx).$$

Another check.

```
>> diff(int_y1,x)
ans = -a/b/(a+b*x)+1/b ...
```

$$= \frac{1}{b} - \frac{ab}{b^2(a+bx)} = \frac{a+bx-a}{b(a+bx)} = \frac{x}{a+bx}.$$

Now we can come back to our initial task

Integrating Eq. (c)

```
clear
syms a0 k v v0
y1 = v/(a0 + k*v);
int_y1 = int(y1,v)
upper = subs(int_y1,v,v0/2)
lower = subs(int_y1,v,v0)
res1 = upper - lower;
pretty(res1)
```

we get the unknown distance in the form

```
       v0    a0 log(a0 + 1/2 k v0)   a0 log(a0 + k v0)
- 1/2 ---- - --------------------- + -----------------  .
       k                2                     2
                       k                     k
```

Example – a falling particle influenced by air resistance

Given : h, v_0, we assume that $a = g - kv^2$.

The acceleration of a particle falling in the vicinity of the Earth's surface can be approximated by an experimentally obtained relation, namely that the acceleration proportionally decreases with respect to the square of immediate velocity. This is actually a dynamic task, treated by a so-called phenomenological approach based on results of observation. See Fig. K02.

Determine: Hit velocity v_k and time to hit the ground, i.e. T.

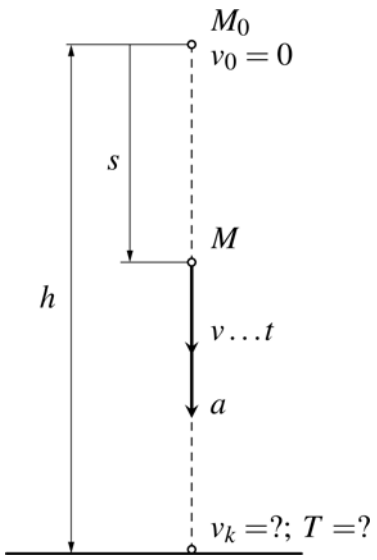


Fig. K02. A falling particle in the air

The velocity as a function of the distance $v = v(s)$ is obtained by the following procedure

$$\frac{v\mathrm{d}v}{\mathrm{d}s} = g - kv^2, \qquad \frac{v\mathrm{d}v}{g - kv^2} = \mathrm{d}s, \qquad \int_0^v \frac{v\mathrm{d}v}{g - kv^2} = \int_0^s \mathrm{d}s.$$

Matlab helps again

```
int(v/(g-k*v^2),v)=

                          2
        log(-g + k v )
- 1/2 --------------
              k
```

So,

$$-\left[\frac{\lg(-g + kv^2)}{2k}\right]_0^v = s, \left[\frac{\lg(-g + kv^2)}{2k} - \frac{\lg(-g)}{2k}\right] = -s, \ \lg(-g + kv^2) - \lg(-g) = -2ks,$$

$$\lg\frac{-g + kv^2}{-g} = -2ks, \quad \frac{-g + kv^2}{-g} = \mathrm{e}^{-2ks}, -g + kv^2 = -g\mathrm{e}^{-2ks}, \ kv^2 = g - g\mathrm{e}^{-2ks} = g(1 - \mathrm{e}^{-2ks}).$$

And finally, we get the velocity as a function of the distance

$$v = \sqrt{\frac{g}{k}\left(1 - \mathrm{e}^{-2ks}\right)}.$$

The velocity as a function of time $v = v(t)$ is obtained from

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - kv^2,$$

$$\frac{\mathrm{d}v}{g-kv^2}=\mathrm{d}t\,,$$

$$\int_0^v \frac{\mathrm{d}v}{g-kv^2}=\int_0^t \mathrm{d}t\,.$$

Matlab gives `int(1/(g-k*v^2),v)= 1/(g*k)^(1/2)*atanh(k*v/(g*k)^(1/2))`.

So, on the left hand of the equation we have

$$\int \frac{\mathrm{d}v}{g-kv^2}=\frac{\operatorname{arctanh}\dfrac{kv}{\sqrt{gk}}}{\sqrt{gk}}\,.$$

Example – a particle in gravitational field

Given: R,h,v_0,k. The Newton's gravitational law states that the gravitational acceleration at a distance x from the centre of the Earth is inversely proportional to the square of that distance

$$a=-\frac{k}{x^2}\,. \qquad \text{(a)}$$

Determine: The velocity of a particle being shot upwards, from a location at the height h above the surface, by velocity v_0, as a function of the distance s, which is measured from the surface of the Earth. See Fig. K03.

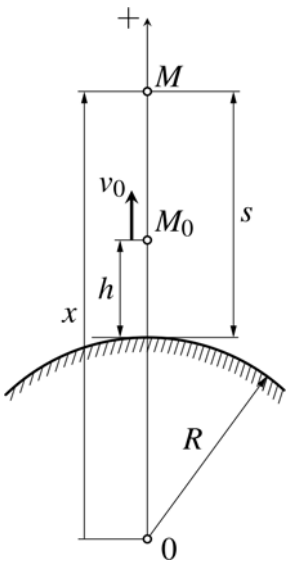


Fig. K03. A particle in gravitational field

Integrating (a)

$$\frac{v\mathrm{d}v}{dx}=-\frac{k}{x^2}\,,$$

$$\int_{v_0}^{v} v\mathrm{d}v=-k\int_{R+h}^{R+s}\frac{\mathrm{d}x}{x^2}\,,\qquad \left[\frac{v^2}{2}\right]_{v_0}^{v}=-k\left[-\frac{1}{x}\right]_{R+h}^{R+s}\,,$$

$$v^2=v_0^2+2k\left(\frac{1}{R+s}-\frac{1}{R+h}\right).$$

So

$$v=\sqrt{v_0^2+2k\left(\frac{1}{R+s}-\frac{1}{R+h}\right)}\,.$$

Example – minimization task

Given: h, b, c_1, c_2. There are two locations A and B. See Fig. K04. The x axis represents a paved road. One can travel along that road by the velocity c_1. Outside of the road there is a rough terrain where one can ride more slowly by the velocity $c_2 < c_1$.

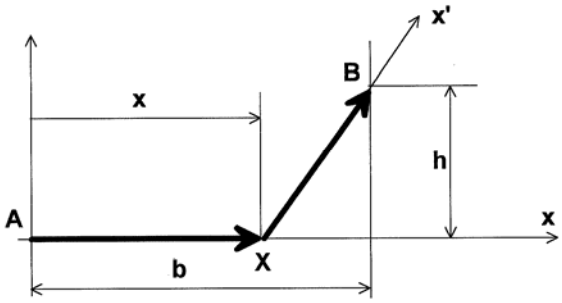


Fig. K04. Road field trip

Determine: The location x, where one should leave to road, and then proceed directly, by a straight line, to the point B in order to minimize the travel time between A and B.

Assuming that both velocities are constant, one can write

$$t = t_1 + t_2 = \frac{x}{c_1} + \frac{\sqrt{h^2 + (b-x)^2}}{c_2}.$$

To find the extremum we compute a derivative of the above relation with respect to x

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{c_1} + \frac{1}{c_2}\frac{1}{2}\frac{2(b-x)(-1)}{\sqrt{h^2 + (b-x)^2}}$$

and then equal it to zero

$$\frac{1}{c_1} - \frac{1}{c_2}\frac{(b-x)}{\sqrt{h^2 + (b-x)^2}} = 0.$$

Solving it for x we get two roots with opposite signs. Taking the positive value only we get

$$x = b - \frac{c_2 h}{\sqrt{c_1^2 - c_2^2}}.$$

It should be reminded that the above formulas are valid under two limiting conditions, namely, $c_1 > c_2$ and $\frac{b}{h} > \frac{c_2}{\sqrt{c_1^2 - c_2^2}}$.

Choosing $b = h = 2\,\mathrm{km}$ we might obtain the following table of x values (expressed in [km]) for different combinations of velocities c_1, c_2 in [km/hour] needed to minimize the total travel time.

		c2						
		10	20	30	40	50	60	70
	20	0.8453	*	*	*	*	*	*
	30	1.2929	0.2111	*	*	*	*	*
	40	1.4836	0.8453	*	*	*	*	*
	50	1.5918	1.1271	0.5000	*	*	*	*
c1	60	1.6619	1.2929	0.8453	0.2111	*	*	*
	70	1.7113	1.4037	1.0513	0.6074	*	*	*
	80	1.7480	1.4836	1.1910	0.8453	0.3987	*	*
	90	1.7764	1.5442	1.2929	1.0077	0.6637	0.2111	*
	100	1.7990	1.5918	1.3710	1.1271	0.8453	0.5000	0.0396

The overall **time to destination** [in hours] for different combinations of velocities c_1, c_2 is

		c2						
		10	20	30	40	50	60	70
	20	0.2732	*	*	*	*	*	*
	30	0.2552	0.1412	*	*	*	*	*
	40	0.2436	0.1366	*	*	*	*	*
	50	0.2360	0.1317	0.0933	*	*	*	*
c1	60	0.2305	0.1276	0.0911	0.0706	*	*	*
	70	0.2265	0.1244	0.0888	0.0696	*	*	*
	80	0.2234	0.1218	0.0868	0.0683	0.0562	*	*
	90	0.2210	0.1197	0.0851	0.0670	0.0555	0.0471	*
	100	0.2190	0.1180	0.0836	0.0658	0.0546	0.0467	0.0404

See the program `K02_time_to_destination`.

```
%K02_time_to destination
clear

z = zeros(10,10);
t = zeros(10,10);
c1_to_c2 = zeros(10,10);

c1 = 10:10:100;
c2 = 10:10:100;
b = 2; h = 2;
for i = 1:10
  for j = 1:i-1
    if (c1(i) > c2(j)),
        c1_to_c2(i,j) = 1;
    end
    b_to_h(i,j) = c2(i)/sqrt(c1(i)^2 - c2(j)^2);
% distance x
    z(i,j)= b - c2(j)*h/sqrt(c1(i)^2 - c2(j)^2);
% compute time to destination
    t(i,j) = z(i,j)/c1(i) + sqrt(h^2 + (b - z(i,j))^2)/c2(j);
  end
end

sz_z = size(z)
sz_t = size(t)

% take only positive members of z
for i=1:10
  for j=1:10
    if z(i,j)<0, z(i,j)=0; end;
    if t(i,j)<0, t(i,j)=0; end;
  end
 end

z;
t;
b_to_h
c1_to_c2

%%%%%%%%%%%%%%%%%%%%%%
```

```
b = 2; h = 2; cc1 = 20; cc2 = 10;

bb_to_hh = cc2/sqrt(cc1^2 - cc2^2);
xx = b - cc2*h/sqrt(cc1^2 - cc2^2)
disp([cc1 cc2])
z

figure(1);
map=[0.8 0.8 0.8];
colormap(map);
subplot(1,2,1); surf(c1,c2,z); ; grid; view(30,30)
subplot(1,2,2); surf(c1,c2,t);
view(30,30);xlabel('velocity c2');
ylabel('velocity c1'); title('distance x'); grid
print -djpeg -r300 fig_k4_c1

% end of k4_c2
```

The contribution of high velocities is small. This can also be documented on a trivial example. Consider a distance composed of two identical parts, say s, and assume that a car travels the first part by velocity v_1 while the second part by velocity v_2. The time for the first part is $t_1 = s/v_1$. For the second part it is $t_2 = s/v_2$. The overall time to destination is $t = t_1 + t_2$. Then, the corresponding average velocity is

$$v_{\text{avg}} = \frac{2s}{t_1 + t_2} = \frac{2s}{s/v_1 + s/v_2} = \frac{2v_1 v_2}{v_1 + v_2}.$$

So, for $v_1 = 100$ km/h and $v_2 = 1$ km/h we get 1.9802 km/h.

K2.2. Motion along a curve

In Fig. K05 the particle L (sometimes we say the point L) is constrained to the spatial curve k_L. We say that the particle follows the curve k_L with the velocity $\vec{v}$. Presently, it has the acceleration $\vec{a}$. The velocity vector lies in the tangent line, the acceleration vector is confined to the plane formed by tangent and normal lines. The curve k_L is called a trajectory of the particle L. The tangent line t, the normal line n and the binormal line b determine the immediate basic kinematic orientation of the particle L.

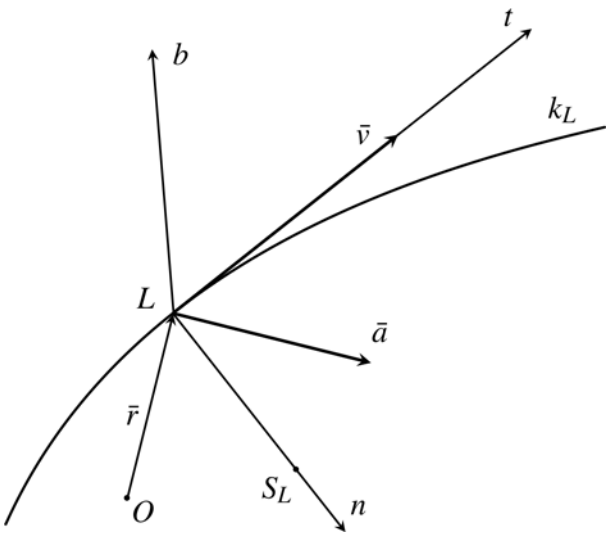


Fig. K05. Triple of normals

The motion of the particle is determined by its radius vector whose dependence on time is known.

$$\vec{r} = \vec{r}(t). \qquad \text{(K\_11)}$$

The velocity and acceleration are

$$\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}, \qquad \vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}. \tag{K\_12, 13}$$

The motion of a particle is then described either in the vector notation by

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \tag{K\_14}$$

or in the scalar notation (scalar equations actually represents parametrical equations of the trajectory) by

$$x = x(t), \quad y = y(t), \quad z = z(t). \tag{K\_15}$$

Arc length, measured from the initial position of the particle at $t = t_0$, is

$$s = \int_{t_0}^{t} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,\mathrm{d}t. \tag{K\_16}$$

Velocity

$v_x = \dot{x}(t), \quad v_y = \dot{y}(t), \quad v_z = \dot{z}(t)$ … velocity components, (K_17)

$v = |\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$ … magnitude of velocity, speed, (K_18a)

$\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$ … velocity vector. (K_18b)

Acceleration

$a_x = \dot{v}_x(t), \quad a_y = \dot{v}_y(t), \quad a_z = \dot{v}_z(t)$ … acceleration components, (K_19)

$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$ … magnitude of acceleration, (K_20)

$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$ … acceleration vector. (K_21)

Acceleration lies in the osculating plane being formed by normal and tangent lines. It could also be decomposed into normal and tangent components. For magnitudes we write

$$a = \sqrt{a_t^2 + a_n^2},$$

where

$a_t = \dot{v} = \ddot{s} = \frac{v\mathrm{d}v}{\mathrm{d}s} = \frac{\mathrm{d}(v^2)}{2\mathrm{d}s}$ ... tangential acceleration, (K_22)

$a_n = \frac{v^2}{\rho}$ ... normal or centripetal acceleration, (K_23)

ρ ... curvature radius. (K_24)

We could also write

$$a_x = \dot{v}_x = \ddot{x} = \frac{\mathrm{d}v_x^2}{2\mathrm{d}x} = \frac{v_x \mathrm{d}v_x}{\mathrm{d}x}, \tag{K_25}$$

$$a_y = \dot{v}_y = \ddot{y} = \frac{\mathrm{d}v_y^2}{2\mathrm{d}y} = \frac{v_y \mathrm{d}v_y}{\mathrm{d}y}, \tag{K_26}$$

$$a_z = \dot{v}_z = \ddot{z} = \frac{\mathrm{d}v_z^2}{2\mathrm{d}z} = \frac{v_z \mathrm{d}v_z}{\mathrm{d}z}. \tag{K_27}$$

Example – motion along an ellipse

Given: The motion is expressed by $x = x_0 \cos\omega t, y = y_0 \sin\omega t$, where ω is so-called angular frequency.
Determine: Velocity and acceleration components.

$$\dot{x} = -x_0\omega \sin\omega t, \quad \dot{y} = y_0\omega \cos\omega t,$$
$$\ddot{x} = -x_0\omega^2 \cos\omega t, \quad \ddot{y} = -y_0\omega^2 \sin\omega t.$$

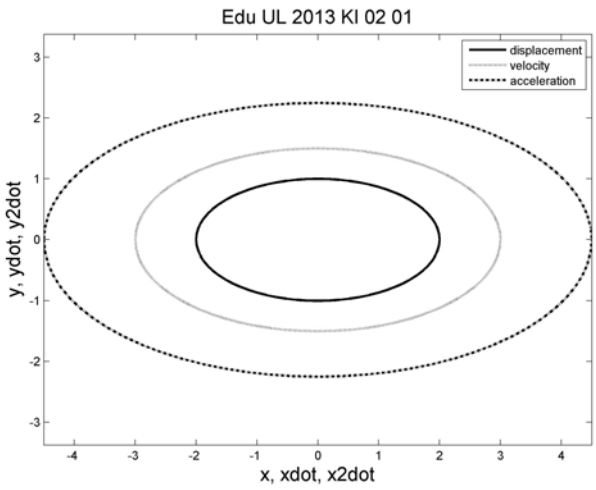


Fig. K06. Displacement, velocity and acceleration

See the program `K03_motion of a particle along an ellipse` and Fig. K06.

```
% K03_motion of a particle along an ellipse
% original file name is Edu_UL_2013_KI_02_01
clear
omt = 0:pi/36:2*pi;
omega = 1.5;
x0 = 2; y0 = 1;
x = x0*cos(omt); y = y0*sin(omt);
xdot = -x0*omega*sin(omt); ydot = y0*omega*cos(omt);
x2dot = -x0*omega^2*cos(omt); y2dot =  -y0*omega^2*sin(omt);
figure(1)
plot(x,y,'k-', xdot,ydot,'k:', x2dot,y2dot,'k-.', 'linewidth', 2)
title('Edu UL 2013 KI 02 01', 'fontsize', 16)
legend('displacement', 'velocity', 'acceleration', 'fontsize', 16)
```

```
xlabel('x, xdot, x2dot', 'fontsize', 16);
ylabel('y, ydot, y2dot', 'fontsize', 16);
print -djpeg -r300 fig_KI_02_01
```

Example – particle motion composed of rotation and translation

Given: The rod, see Fig. K07, rotates at a constant angular velocity ω. Along the rod, a sleeve – simplified as a particle M – slides by a constant velocity c.
Determine: The particle's displacement, velocity and acceleration of the point M as functions of time. At the beginning the rod was coincident with x axis, i.e. $\varphi = 0$, and the initial location of M was defined by the distance l from the origin O.

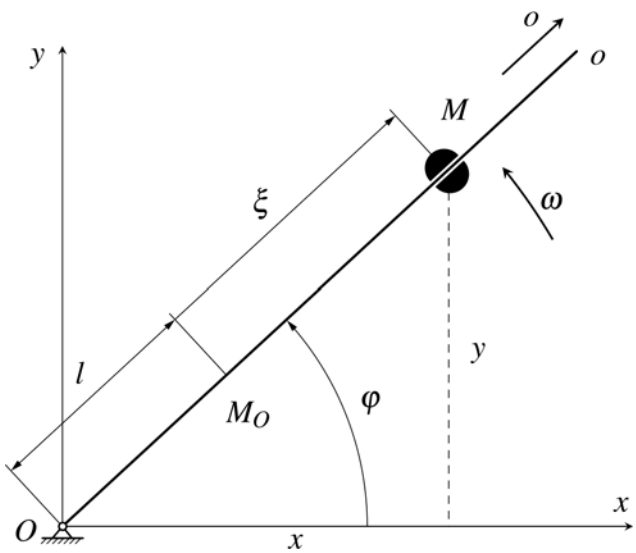


Fig K07. The sleeve on a rotating rod

The coordinates of the point M are

$$x = (l + \xi)\cos\varphi, \quad y = (l + \xi)\sin\varphi.$$

Due to our assumptions concerning constant velocities we get

$$\xi = ct, \quad \varphi = \omega t\,.$$

So

$$x = (l + ct)\cos\omega t, \quad y = (l + ct)\sin\omega t.$$

Velocity

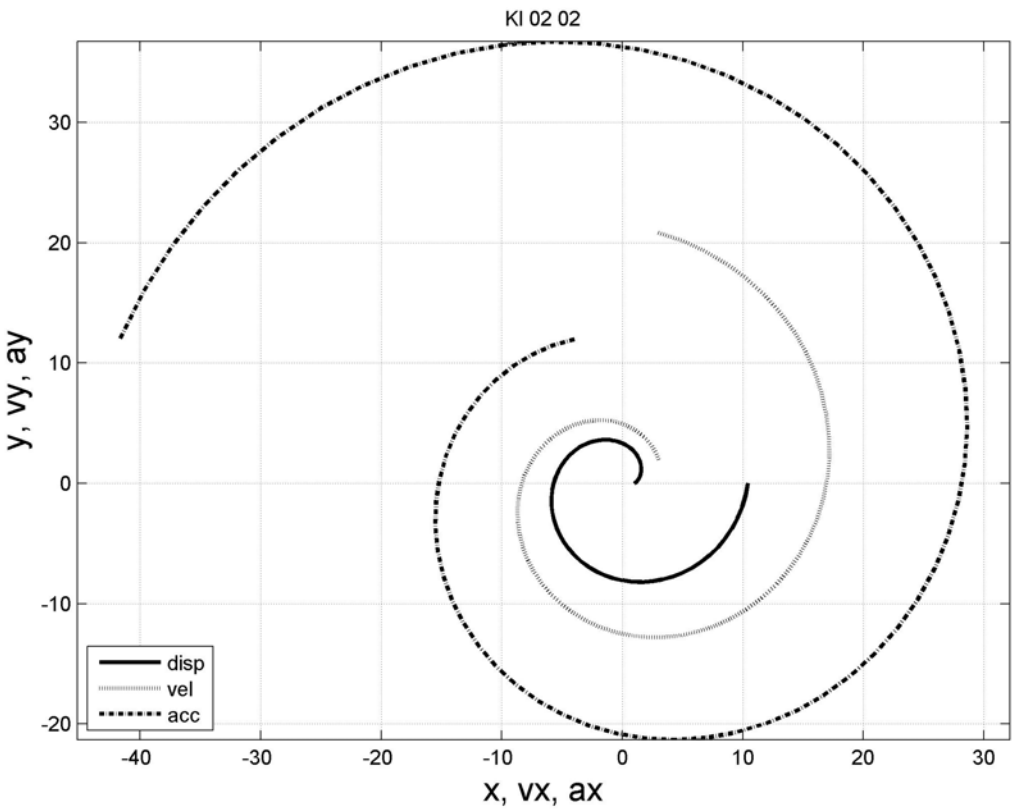
$$v_x = \frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = c\cos\omega t + (l + ct)\omega(-\sin\omega t) = c\cos\omega t - \omega(l + ct)\sin\omega t\,,$$

$$v_y = \frac{\mathrm{d}y}{\mathrm{d}t} = \dot{y} = c\sin\omega t + (l + ct)\omega\cos\omega t = c\sin\omega t + \omega(l + ct)\cos\omega t\,.$$

Acceleration

$$a_x = \frac{\mathrm{d}v_x}{\mathrm{d}t} = \dot{v}_x = \ddot{x} = -c\omega\sin\omega t - c\omega\sin\omega t - \omega^2(l + ct)\cos\omega t = -2c\omega\sin\omega t - \omega^2(l + ct)\cos\omega t\,,$$

$$a_y = \frac{\mathrm{d}v_y}{\mathrm{d}t} = \dot{v}_y = \ddot{y} = c\omega\cos\omega t + c\omega\cos\omega t - \omega^2(l + ct)\sin\omega t = 2c\omega\cos\omega t - \omega^2(l + ct)\sin\omega t\,.$$



Fig, K08. Displacements, velocities and accelerations

See the program `K03_rotation and translation.m` and Fig. K08.

```
% K03_rotation and translation
% original file name is Edu_UL_2013_KI_02_02
clear
l = 1; om = 2; c = 3;
t = 0:pi/64:pi;
len = length(t);
t_ones = ones(1,len);
x = (l*t_ones + c*t).*cos(om*t);
y = (l*t_ones + c*t).*sin(om*t);
vx = c*cos(om*t) - om*(l*t_ones + c*t).*sin(om*t);
vy = c*sin(om*t) + om*(l*t_ones + c*t).*cos(om*t);
ax = -2*c*om*sin(om*t) - om^2*(l*t_ones + c*t).*cos(om*t);
ay =  2*c*om*cos(om*t) - om^2*(l*t_ones + c*t).*sin(om*t);
figure(1)
plot(x,y,'k-', vx,vy,'k:', ax,ay,'k-.', 'linewidth', 2);
grid; axis('equal')
legend('disp', 'vel', 'acc',3)
xlabel('x, vx, ax', 'fontsize', 16)
ylabel('y, vy, ay', 'fontsize', 16)
title('KI 02 02')
print -djpeg -r300 fig_KI_02_02
```

K3. Rotary and translatory motion of bodies

K3.1. Rotary motion of a body

K3.1.1. Scalar approach

The body is subjected to a rotary motion if one of its material lines (such a line is called the axis of rotation or the rotation axis) always stays in rest. The trajectories of all the body's particles are circles lying in planes perpendicular to the axis of rotation and having their centers at the axis of rotation.

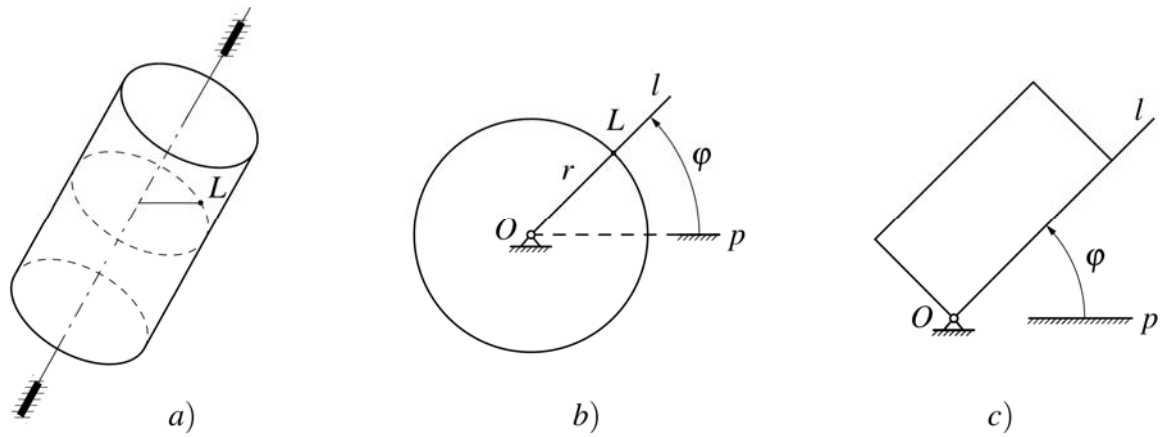


Fig. K09. Rotation

When solving planar problems, the axis of rotation appears to be a single point viewed from above. The rotary motion can be identified by an angle, say φ, between the radius vector of an arbitrary point and a line being firmly connected to the rigid frame. See Fig. K09. The angle of rotation is usually expressed as a function of time

$$\varphi = \varphi(t)\,. \tag{K\_28}$$

Then, the angular velocity is defined as the time rate of the rotation angle.

$$\omega(t) = \frac{\mathrm{d}\varphi}{\mathrm{d}t}\,. \tag{K\_29}$$

And the angular acceleration is the time rate of angular velocity.

$$\varepsilon(t) = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} = \frac{\mathrm{d}\left(\omega^2\right)}{2\mathrm{d}\varphi} = \frac{\omega\,\mathrm{d}\omega}{\mathrm{d}\varphi}\,. \tag{K\_30}$$

The angle φ is usually measured in radians [1]. So, the angular velocity and acceleration are measured in $[\text{radians}/\text{s}]$ and $[\text{radians}/\text{s}^2]$. Since the radian, as the measure of an angular distance, is a dimensionless value, the above units are frequently expressed by $[1/\text{s}]$ and $[1/\text{s}^2]$, respectively.

Rotary 'speed' is often measured by counting the number of *revolutions per minute*, known under the abbreviation R.P.M. Realizing that one revolution equals the angle of 2π and that there are 60 seconds into one minute, one can simply deduce that

$$\omega\,[1/\mathrm{s}] = \pi\, n/30\,[\text{revolutions per minute}], \qquad \text{(K\_31)}$$

where we denoted the quantity revolutions per minute by a symbol n .

K3.1.2. Vector approach is more general

The angular velocity and angular acceleration are actually vectors, denoted $\vec{\omega}, \vec{\varepsilon}$, whose lines of actions are identical with the axis of rotation, say o , defined by angles (α, β, γ). See Fig. K10. One can write

$$\vec{\omega} = \vec{\omega}_x + \vec{\omega}_y + \vec{\omega}_z = \omega_x \vec{i} + \omega_y \vec{j} + \omega_k \vec{k}\,, \qquad \text{... (K\_32a)}$$

$$\vec{\varepsilon} = \vec{\varepsilon}_x + \vec{\varepsilon}_y + \vec{\varepsilon}_z = \varepsilon_x \vec{i} + \varepsilon_y \vec{j} + \varepsilon_k \vec{k}\,. \qquad \text{... (K\_32b)}$$

Denoting unit vectors by $\vec{i}, \vec{j}, \vec{k}$ we can express the velocity of a generic point, say L, by the vector product

$$\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix}. \qquad \text{(K\_33)}$$

The acceleration of the point L is obtained by the derivative of the above relation with respect to time

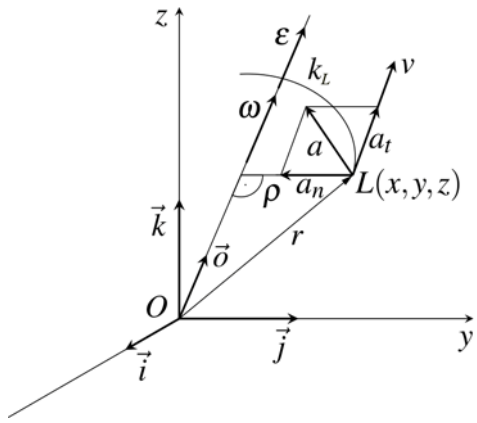


Fig. K10. Vectors of velocities and accelerations

$$\vec{a} = \frac{\mathrm{d}(\vec{\omega} \times \vec{r})}{\mathrm{d}t} = \frac{d\vec{\omega}}{\mathrm{d}t} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{\mathrm{d}t} = \underbrace{\vec{\varepsilon} \times \vec{r}}_{\vec{a}_t} + \underbrace{\vec{\omega} \times \vec{v}}_{\vec{a}_n}. \qquad \text{(K\_34)}$$

The acceleration components are known as

the tangent acceleration $\vec{a}_t = \vec{\varepsilon} \times \vec{r}$ and (K_35)

the normal, or centripetal, acceleration $\vec{a}_n = \vec{\omega} \times \vec{v} = \vec{\omega} \times \vec{\omega} \times \vec{r}$. (K_36)

Decomposing the relation $\vec{a} = \underbrace{\vec{\varepsilon} \times \vec{r}}_{\vec{a}_t} + \underbrace{\vec{\omega} \times \vec{v}}_{\vec{a}_n}$ into Cartesian components we get

$$a_x = \varepsilon_y z - \varepsilon_z y + \omega_y v_z - \omega_z v_y\,, \qquad \text{(K\_37a)}$$

$$a_y = \varepsilon_z x - \varepsilon_x z + \omega_z v_x - \omega_x v_z\,, \qquad \text{(K\_37b)}$$

$$a_z = \varepsilon_x y - \varepsilon_y x + \omega_x v_y - \omega_y v_x\,. \qquad \text{(K\_37c)}$$

It is of interest to analyze a special case, i.e. the motion of the point, say L, along a circle with the radius r.

The immediate Cartesian coordinates of the point L can be expressed by

$$x = r\cos\varphi,\ y = r\sin\varphi, \qquad \text{(K\_38)}$$

where the angle φ, indicating the immediate angular position of the point L, is a function of time and is measured from the x axis counterclockwise.

Generally, the angle φ depends on the angular velocity and the angular velocity depends on time, namely $\varphi = f(\omega), \omega = g(t)$. To express the Cartesian components of velocity and acceleration of the point L as functions of time, we have to evaluate the first and second derivatives of Eq. (K_38). Thus

$$v_x = -r\omega\sin\varphi = -\omega y, \qquad \text{(K\_38a)}$$
$$v_y = +r\omega\cos\varphi = +\omega x. \qquad \text{(K\_39b)}$$

$$a_x = -r\omega^2\cos\varphi - r\varepsilon\sin\varphi = -\omega^2 x - \varepsilon y, \qquad \text{(K\_40a)}$$
$$a_y = -r\omega^2\sin\varphi + r\varepsilon\cos\varphi = -\omega^2 y + \varepsilon x. \qquad \text{(K\_40b)}$$

The above relations are simplified if $\omega = const$, because it that case $\varepsilon = 0$.

Often, the analysis is provided using not Cartesian but polar components, that are defined in tangent (t) and normal (n) directions. For magnitudes vector quantities $\vec{v}, \vec{a}$ we could write

$s = r\varphi$	... arc displacement measured along the circle,	(K_41)
$v = r\omega$	... velocity which has always the tangential direction,	(K_42)
$a_{\mathrm{t}} = r\varepsilon$	... tangential component of acceleration,	(K_43)
$a_{\mathrm{n}} = r\omega^2$	... normal, or centripetal, component of acceleration,	(K_44)
$a = r\sqrt{\omega^4 + \varepsilon^2}$.	… magnitude of resulting acceleration $\vec{a} = \vec{a}_t + \vec{a}_n$.	(K_45)

K3.2. Harmonic motion

The term harmonic motion is frequently used in mechanics. Imagine that you project a radius vector $\vec{r}$, rotating counterclockwise by a constant angular velocity ω, into the vertical coordinate axis and then subsequently register the obtained values as a function of time.
See Fig. K11 , the program `K04_harmonic_motion` and Fig. K12.

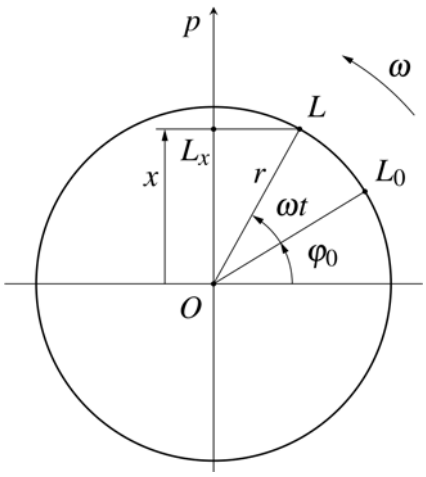


Fig. K11. Rotating radius

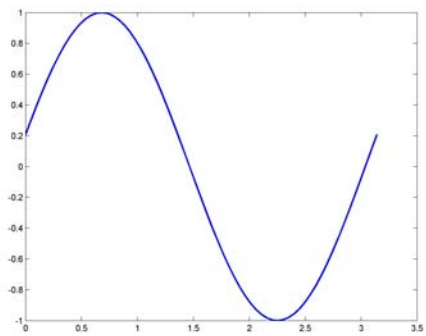


Fig. K12. Harmonic motion

```
% K04_harmonic_motion
clear; omega1 = 2; omega2 = 6; t = 0:pi/124:pi; t0 = pi/15;
y1 = sin(omega1*t + t0); y2 = sin(omega2*t);
figure(1)
plot(t,y1, 'linewidth', 2); print -djpeg -r300 fig_harmonic_motion
figure(2)
plot(y1,y2, 'linewidth', 2); axis([-1.1 1.1 -1.1 1.1])
print -djpeg -r300 fig_lissajouse_motion
```

The harmonic function is most frequently described by a sine or cosine functions of time. In this case, we can write

$$x = r\sin(\omega t + \varphi_0), \qquad \text{(K\_46)}$$

where we define

r	amplitude of motion,
x	immediate displacement,
ω	angular frequency,
t	independent variable, usually time,
φ_0	initial angle, phase.

Note: Composing harmonic motions occurring in two perpendicular directions we get so-called Lissajouse curves. An example for motions, whose frequencies are in the ratio 1:2, we get Fig. K13.

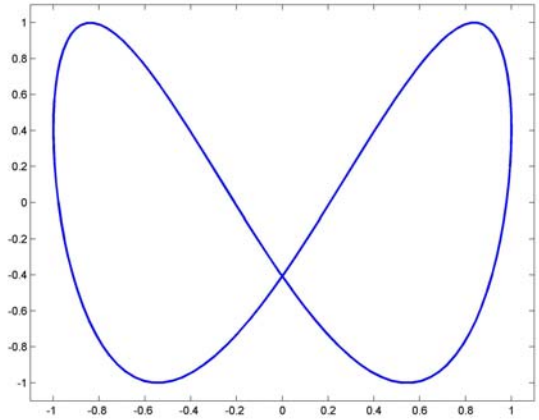


Fig. K13. Lissajouse curve

K3.3. Translatory motion of a body

The body is subjected to a translatory motion if at least two of its nonparallel lines do not change their angles during the rotation. In that case, all the particles of the body move along identical curves. At a given moment the velocities and accelerations of all the body's particles are the same. Of course, in another moment they are different with respect to the previous one.

When analyzing any translatory motion of a body, regardless of considering the motion along a line, or along a planar, or spatial, curves, it suffices to study the motion of a single particle.

Example – translatory motion
Given: The body is attached to the ground by two parallelogram links. See Fig. K14. The members 2 and 4, having the length r, are accelerating with a constant angular acceleration $\varepsilon = k$.
Determine: The trajectory, velocity and acceleration of the point T.

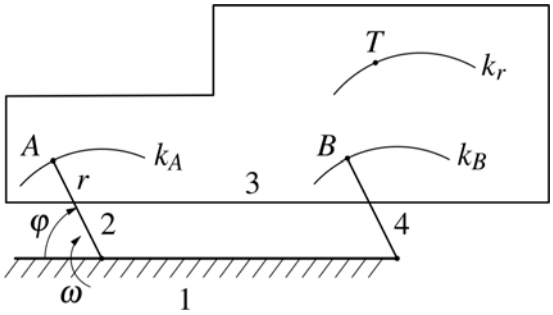


Fig. K14. Translatory motion

Since the body 3 is subjected to a translatory motion, all the particles follow the same trajectory, i.e. the same circles, as the particle A.

Assuming the initial conditions as $t = 0, \varphi = 0, \omega = 0$, the velocity of the particle A is the same as that of other particles of that body. So, the magnitudes of velocities are

$$v = v_{\mathrm{T}} = v_{\mathrm{A}} = r\omega = rkt,$$

and consequently

$$\omega = \varepsilon t = kt.$$

Magnitudes of normal components of all the points is $a_{\mathrm{n}} = a_{\mathrm{nT}} = a_{\mathrm{nA}} = r\omega^2 = r(kt)^2$.
Magnitudes of tangential components of all the points is $a_{\mathrm{t}} = a_{\mathrm{tT}} = a_{\mathrm{tA}} = r\varepsilon = rk$.
Magnitudes of the resulting accelerations is $a = \sqrt{a_{\mathrm{t}}^2 + a_{\mathrm{n}}^2} = rk\sqrt{1 + k^2 t^4}$.
Of course, the directions of all the vectors vary, as they travel along the circles.

K4. Acceleration of a particle in a non-inertial frame of reference

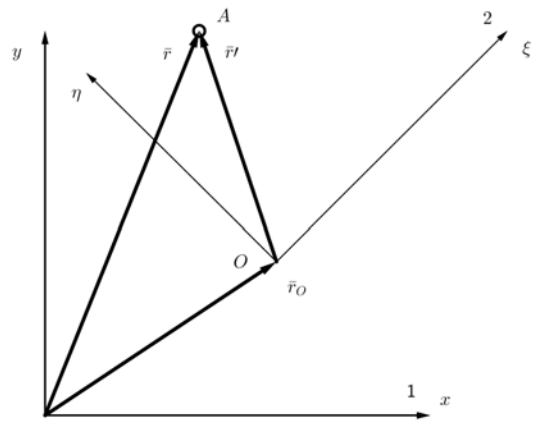


Fig. K15. Kinematics of relative motions

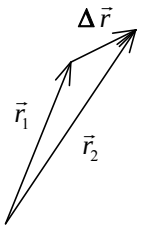
Consider an inertial coordinate system (x, y) labeled 1 in Fig. K15. In this coordinate system, there is another system (ξ, η), labeled 2. The position of the origin O of the system (ξ, η) is determined by the vector $\vec{r}_O$. The origin O has the velocity $\vec{v}_O$. The system (ξ, η) moves with respect to the system (x, y) and also rotates around the origin O with the angular velocity $\vec{\omega}$ and with the angular acceleration $\vec{\varepsilon}$. The point A, lying in the coordinate system (ξ, η), moves as well. Its position with respect to the coordinate system (x, y) is defined by the vector $\vec{r}$, while the position with respect to the system (ξ, η) is defined by the vector $\vec{r}'$. The coordinate system (ξ, η) is obviously non-inertial.

The time derivative of the angular velocity $\vec{\omega}$, i.e. the angular acceleration $\vec{\varepsilon}$, is independent of the choice of the coordinate system. So

$$\left[\frac{\mathrm{d}\vec{\omega}_{21}}{\mathrm{d}t}\right]_1 = \left[\frac{\mathrm{d}\vec{\omega}_{21}}{\mathrm{d}t}\right]_2 = \vec{\varepsilon}_{21}. \qquad \text{(K\_47)}$$

For the positional vectors we can write

$$\vec{r} = \vec{r}_O + \vec{r}'. \qquad \text{(K\_48)}$$



The time derivative of a positional vector $\vec{r}$ is defined as a vector having the direction of the trajectory of the motion of a point the positional vector is pointing to. See Fig. K16. In the limit, we have

Fig. K16. Time derivative of a vector

$$\lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{v}\,. \qquad \text{(K\_49)}$$

So, the time derivative of a vector is the velocity of its end point. The velocity of a point A, with respect to the coordinate system 1, is given by the time derivative of the positional vector $\vec{r}$ in that system, i.e. (x, y), which has to be equal to the sum of time derivatives of vectors $\vec{r}_O$ a $\vec{r}\,'$ in the same system, thus

$$\vec{v}_{\mathrm{A}} = \left[\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right]_1 = \left[\frac{\mathrm{d}\vec{r}_O}{\mathrm{d}t}\right]_1 + \left[\frac{\mathrm{d}\vec{r}\,'}{\mathrm{d}t}\right]_1 = \quad \text{… the first term is the velocity of the point O}$$

$$= \vec{v}_O + \left[\frac{\mathrm{d}\vec{r}\,'}{\mathrm{d}t}\right]_2 + \vec{\omega}_{21} \times \vec{r}\,' = \quad \text{… the second term can be expressed as a time derivative}$$

$$= \underbrace{\vec{v}_O + \vec{\omega}_{21} \times \vec{r}\,'}_{\vec{v}_{\mathrm{carrier}}} + \underbrace{\left[\frac{\mathrm{d}\vec{r}\,'}{\mathrm{d}t}\right]_2}_{\vec{v}_{\mathrm{r}}} = \vec{v}_O + \vec{\omega}_{21} \times \vec{r}\,' + \vec{v}_{\mathrm{r}} = \quad \text{… reordering the terms we get}$$

$$= \vec{v}_{\mathrm{carrier}} + \vec{v}_{\mathrm{relative}} = \vec{v}_{\mathrm{carrier}} + \vec{v}_{\mathrm{r}}\,. \qquad \text{(K\_50)}$$

Let's define

$\vec{v}_{\mathrm{carrier}} = \vec{v}_O + \vec{\omega}_{21} \times \vec{r}\,'$ – carrier velocity and $\vec{v}_{\mathrm{r}} = \left[\frac{\mathrm{d}\vec{r}\,'}{\mathrm{d}t}\right]_2$ – relative velocity.

The acceleration of a point A can be derived similarly. We start with

$$\vec{v}_{\mathrm{A}} = \vec{v}_O + \vec{v}_{\mathrm{r}} + \vec{\omega}_{21} \times \vec{r}\,'. \qquad \text{(K\_51)}$$

Observing the rules for derivatives of products we arrive at

$$\vec{a}_{\mathrm{A}} = \left[\frac{\mathrm{d}\vec{v}_{\mathrm{A}}}{\mathrm{d}t}\right]_1 = \underbrace{\left[\frac{\mathrm{d}\vec{v}_O}{\mathrm{d}t}\right]_1}_{\vec{a}_O} + \underbrace{\left[\frac{\mathrm{d}\vec{v}_r}{\mathrm{d}t}\right]_1}_{\left[\frac{\mathrm{d}\vec{v}_{\mathrm{r}}}{\mathrm{d}t}\right]_2 + \vec{\omega}_{21} \times \vec{v}_{\mathrm{r}}} + \underbrace{\left[\frac{\mathrm{d}\vec{\omega}_{21}}{\mathrm{d}t}\right]_1 \times \vec{r}\,'}_{\vec{\varepsilon}_{21} \times \vec{r}} + \underbrace{\vec{\omega}_{21} \times \left[\frac{d\vec{r}\,'}{dt}\right]_1}_{\vec{\omega}_{21} \times \left\{\left[\frac{d\vec{r}\,'}{dt}\right]_2 + \vec{\omega}_{21} \times \vec{r}\,'\right\}} =$$

$$= \vec{a}_O + \underbrace{\left[\frac{\mathrm{d}\vec{v}_{\mathrm{r}}}{\mathrm{d}t}\right]_2}_{\vec{a}_r} + \vec{\omega}_{21} \times \vec{v}_{\mathrm{r}} + \vec{\varepsilon}_{21} \times \vec{r}\,' + \vec{\omega}_{21} \times \left\{\underbrace{\left[\frac{d\vec{r}\,'}{dt}\right]_2}_{\vec{v}_r} + \vec{\omega}_{21} \times \vec{r}\,'\right\} =$$

$$= \vec{a}_O + \vec{a}_{\mathrm{r}} + \vec{\omega}_{21} \times \vec{v}_{\mathrm{r}} + \vec{\varepsilon}_{21} \times \vec{r}\,' + \vec{\omega}_{21} \times \left(\vec{v}_{\mathrm{r}} + \vec{\omega}_{21} \times \vec{r}\,'\right). \qquad \text{(K\_52)}$$

So, the acceleration of the point A is

$$\boxed{\vec{a}_{\mathrm{A}} = \vec{a}_O + \underbrace{\vec{a}_r}_{\text{relative}} + \underbrace{\vec{\varepsilon}_{21} \times \vec{r}\,'}_{\text{tangential}} + \underbrace{\vec{\omega}_{21} \times \left(\vec{\omega}_{21} \times \vec{r}\,'\right)}_{\text{cenripetal}} + \underbrace{2\vec{\omega}_{21} \times \vec{v}_{\mathrm{r}}}_{\text{Coriolis}}\,. \qquad \text{(K\_53)}}$$

In conclusion, we have derived the acceleration components of a point, subjected to a motion in the non-inertial coordinate system. Evidently, the centrifugal acceleration – incorrectly mentioned in certain textbooks – does not exist. See [1], [3], [5].

K5. Generic motion of bodies in two-dimensional space

In this case, the trajectories, velocities, and accelerations of individual points of the moving body are generally different. However, all the points, lying in lines perpendicular to the plane in which the body lives, have identical trajectories, velocities and acceleration. This is our 2D assumption.

The procedure for analyzing this type of body motion consists of the decomposition of the motion into two parts, named carrier and relative motions, respectively.

Two methods, called the *basic decomposition* and the *Coriolis decompositions*, might alternatively be used.

K5.1. Basic decomposition – the carrier motion is of translatory nature

We start by choosing a suitable reference point – usually, it is a point whose kinematic properties are known. This point becomes an origin of a new coordinate system – we call it a carrier system. Then we make a thought experiment assuming that overall motion of the body is composed of **translatory** motion of the carrier system with respect to the fixed frame of reference plus the relative rotary motion with respect to the carrier system.

The basic frame (Fig. K17) is defined by axes x, y. The carrier frame ξ, η has its origin at the point Ω. Then, the velocity of the point L is

$$\mathrm{L}: \quad \vec{v}_{\mathrm{L}} = \vec{v}_{\Omega} + \vec{v}_{\mathrm{L}\Omega},$$

where its components are

$\vec{v}_{\Omega}$ carrier velocity, which is same as the velocity of the reference point Ω, because we assume that the body is temporarily subjected to a translatory motion only. It is assumed that this velocity is known,

$\vec{v}_{\mathrm{L}\Omega}$ relative velocity of L with respect to the reference point Ω due to the rotary nature of this part of motion.

Similarly for the acceleration of the point L.

$$\mathrm{L}: \quad \vec{a}_{\mathrm{L}} = \vec{a}_{\Omega} + \vec{a}_{\mathrm{L}\Omega}.$$

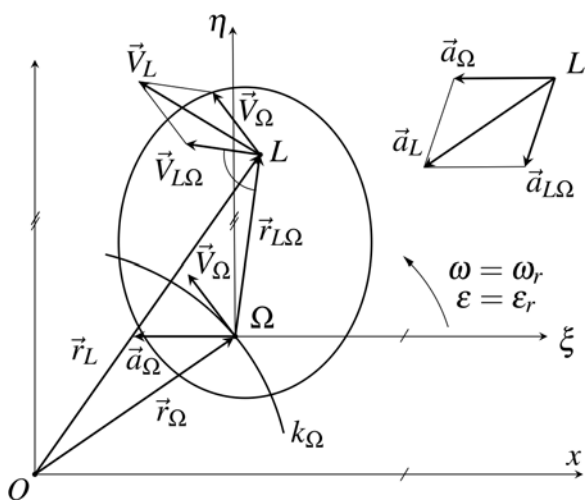


Fig. K17. Decomposition of motions

Now, the *basic decomposition* in more detail.

The location of an arbitrary point L of a body, subjected to a general motion in plane, see Fig. K18, can be described by

$$\vec{r}_{\mathrm{L}} = \vec{r}_{\mathrm{L}\Omega} + \vec{r}_{\Omega}\,, \qquad \text{(K\_54)}$$

where

$\vec{r}_{\mathrm{L}}$ … location of L with respect to the basic frame (x, y),
$\vec{r}_{\mathrm{L}\Omega}$ … location of L with respect to the carrier frame (ξ, η),
$\vec{r}_{\Omega}$ …location of the reference point Ω with respect to the basic frame.

The velocity of the point L is

$$\vec{v}_{\mathrm{L}} = \vec{v}_{\mathrm{L}\Omega} + \vec{v}_{\Omega}\,, \qquad \text{(K\_55)}$$

where

$\vec{v}_{\mathrm{L}\Omega} = \vec{\omega} \times \vec{r}_{\mathrm{L}\Omega}$ … relative velocity,
$\vec{v}_{\Omega}$ … carrier velocity.

The acceleration of the point L is

$$\vec{a}_{\mathrm{L}} = \vec{a}_{\mathrm{L}\Omega} + \vec{a}_{\Omega}\,, \qquad \text{(K\_56)}$$

where $\vec{a}_{\Omega}$ is the carrier component

and the relative acceleration $\vec{a}_{\mathrm{L}\Omega}$ could be decomposed into tangential and normal components as

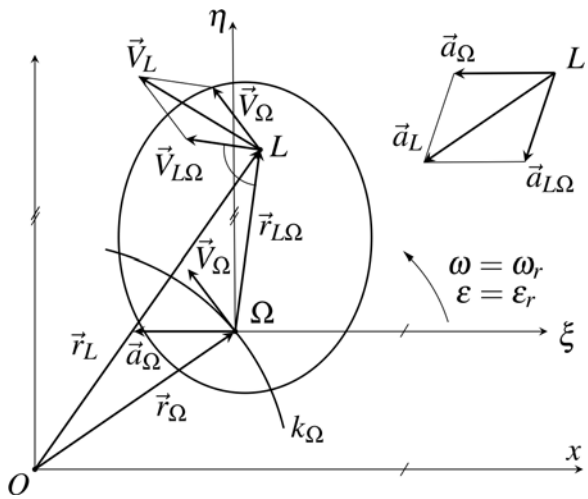


Fig. K18. Basic decomposition

$$\vec{a}_{\mathrm{L}\Omega} = \underbrace{\vec{\varepsilon} \times \vec{r}_{\mathrm{L}\Omega}}_{\vec{a}_{\mathrm{t}}} + \underbrace{\vec{\omega} \times \vec{v}_{\mathrm{L}\Omega}}_{\vec{a}_{\mathrm{n}}}\,. \qquad \text{(K\_57)}$$

The overall angular velocity and overall angular acceleration are identical with the relative angular velocity and with the relative angular acceleration

$$\omega = \omega_{\mathrm{rel}}\,,\ \varepsilon = \varepsilon_{\mathrm{rel}}\,. \qquad \text{(K\_58)}$$

The magnitude of the relative velocity is

$$v_{\mathrm{L}\Omega} = r_{\mathrm{L}\Omega}\omega\,, \qquad \text{(K\_59)}$$

where the magnitudes of above vectors are $v_{\mathrm{L}\Omega} = |\vec{v}_{\mathrm{L}\Omega}|,\ r_{\mathrm{L}\Omega} = |\vec{r}_{\mathrm{L}\Omega}|$. (K_60)

Summary for the basic decomposition. See Fig. K19.

The carrier motion is of translatory nature, so all the points of the considered body have the same velocity and acceleration as the reference point.

The relative motion is of rotary nature. The whole body rotates around the reference point.

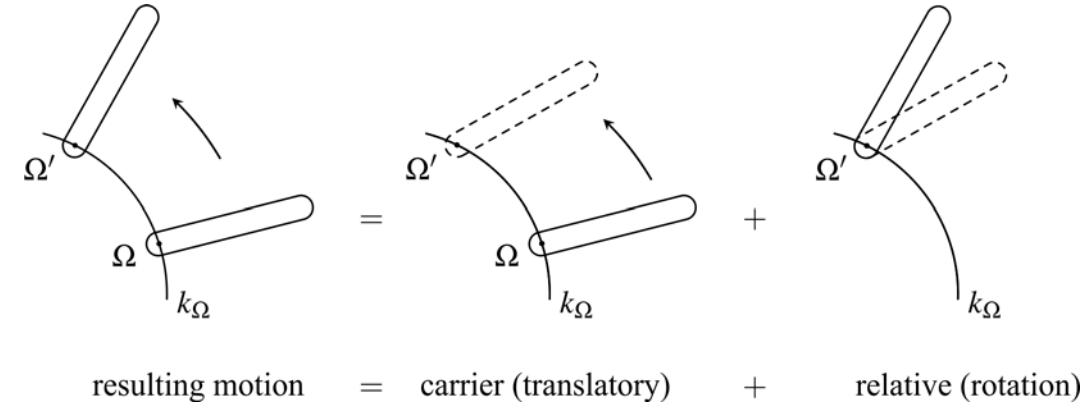


Fig. K19. Basic decomposition

Example – crankshaft mechanism – basic decomposition

Given: The crank mechanism.
Determine: The velocity and the acceleration of the point B belonging to the connecting rod.
Use the basic decomposition depicted in Fig. K20.

The motion (3:1) of the connecting rod (3) with respect to the frame (1) is mentally decomposed into the relative motion (3:5) of the rod (3) with respect to the reference frame (5) plus the translatory motion (5:1) of the reference frame (5) with respect to the basic frame(1). Sometimes, we simply write $31 = 35 + 51$.

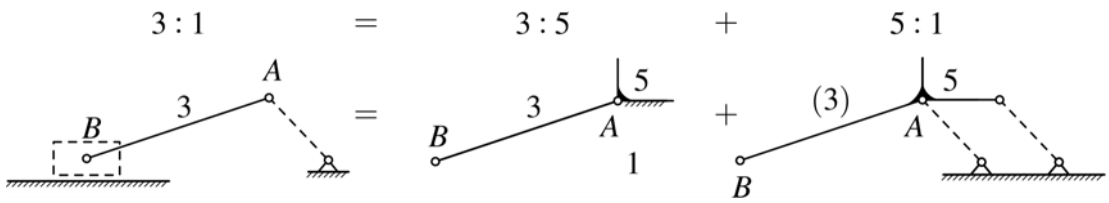


Fig. K20. Basic decomposition for a crankshaft mechanism

The velocity of the point B is

B: $\underset{\rightarrow}{\vec{v}_{31}} = \underset{\rightarrow}{\vec{v}_{35}} + \underline{\vec{v}_{51}}$,

The arrow means that only the direction is known, the underlining means that both direction and magnitude is known.

So,

- the direction of $\vec{v}_{31}$ is known, its magnitude is unknown,
- the direction of $\vec{v}_{35}$ is known, its magnitude is unknown,
- the velocity $\vec{v}_{51} = \vec{v}_{\mathrm{A}}$ is completely known, since the relative motion is of translatory nature.

The acceleration of the point B is

B: $\underset{\longrightarrow}{\vec{a}_{31}} = \underset{\longrightarrow}{\vec{a}_{35}} + \underline{\vec{a}_{51}}$,

where

$\vec{a}_{31}$ … the direction is known,

$\vec{a}_{35}$ … the normal component is known completely, since

$$a_{\mathrm{n}35} = \mathrm{AB}\,\omega^2 \text{, kde } \omega = \frac{v_{35}}{\mathrm{AB}},$$

… while for the tangent component $a_{\mathrm{t}35}$ only the direction is known,

$\vec{a}_{51}$ … completely known since $\vec{a}_{51} = \vec{a}_{\mathrm{A}}$.

So, if a graphic approach were used, the velocity and acceleration could be found easily.

K5.2. Coriolis decomposition – the carrier motion is of rotary nature

Again, we start by choosing a suitable reference point – usually it is a point whose kinematic properties are known. This point becomes an origin of a new coordinate system – we call it the carrier system. Then we make a thought experiment assuming that overall motion of the body is composed of **rotary** motion of the carrier system with respect to the fixed frame plus the relative motion of the body with respect to the carrier system.

Example– crankshaft mechanism – Coriolis decomposition
Given: The crank mechanism
Determine: The velocity and the acceleration of the point B belonging to the connecting rod. Use the Coriolis decomposition depicted in Fig. K21.

The motion (3:1) of the connecting rod (3) with respect to the frame (1) is mentally decomposed into the relative motion (3:2) of the rod (3) with respect to the crank (2) plus the rotary motion (2:1) of the crank (2) with respect to the basic frame (1). Sometimes, we simply write $31 = 32 + 21$.

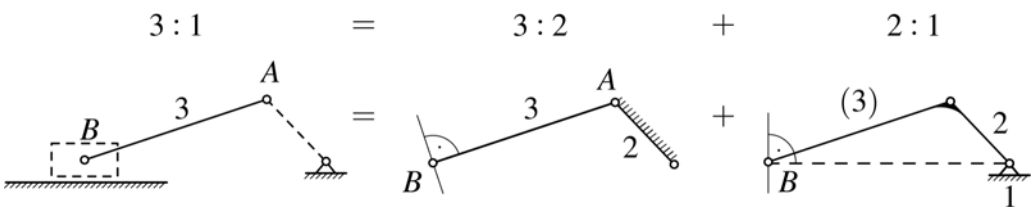


Fig. K21. Coriolis decomposition

The velocity of B is

B: $\underbrace{\vec{v}_{31}}_{\rightarrow} = \underbrace{\vec{v}_{32}}_{\rightarrow} + \underline{\vec{v}_{21}}$, where the magnitude $v_{21} = \mathrm{OB}\,\omega_{21}$ is known.

The acceleration of B is

$\underbrace{\vec{a}_{31}}_{\rightarrow} = \underbrace{\vec{a}_{32}}_{\underline{\mathrm{n}}\ \underline{t}} + \underbrace{\vec{a}_{21}}_{\underline{n}\ t=0} + \underline{\vec{a}_{\mathrm{Cor}}}$, where the magnitudes are $a_{32\mathrm{n}} = \dfrac{v_{32}^2}{\mathrm{BA}}$, $a_{21} = \dfrac{v_{21}^2}{\mathrm{BO}}$ and $a_{\mathrm{Cor}} = 2\omega_{21}v_{32}$.

6. References

[1] Truesdel, C. and Toupin. R.: The classical and field theories. Encyclopedia of Physics, Springer, Berlin, 1960.

[2] Feynman, R.: The Feynman Lectures on Physics, http://en.wikipedia.org/wiki/The_Feynman_Lectures_on_Physics

[3] Stejskal, V., Dehombreux P., Eiber, A., Gupta, R., Okrouhlik, M.: *Mechanics with Matlab*, Electronic Textbook , Faculté Polytechnique de Mons, Belgium, April 2001. ISBN 2-9600226-2-9.
See also http://www.geniemeca.fpms.ac.be

[4] Okrouhlik, M. (editor): *Numerical methods in computational mechanics*. Institute of Thermomechanics, Prague 2009, ISBN 978-80-87012-35-2.
See also http://www.it.cas.cz/files/u1784/Num_methods_in_CM.pdf

[5] Pratap, R. and Ruina, A.: Statics and dynamics, Oxford University Press, 2001.

[6] Kane, T.R.: Dynamics. Holt, Rinehart and Winston, Inc., New York, 1968.

[7] http://en.wikipedia.org/wiki/Kinematics

Dynamics

Scope

1. Introduction to dynamics
2. Dynamics of a particle subjected to a straight line motion
3. Dynamics of a particle subjected to a motion along a curve
4. Dynamics of a particle subjected to a circular motion
5. Newton's and d'Alembert's formulations of equations of motion
6. Vibrations
7. Moments of inertia and deviatoric moments
8. Dynamics of rigid bodies
    - Translatory motion
    - Rotary motion
        - Planar rotary motion
        - Spatial rotary motion about an axis
    - Planar general motion
    - Summary to dynamics of rigid bodies
9. References

D1. Introduction to dynamics

The text is devoted to Newtonian mechanics that is valid for small velocities – small with respect to the speed of light. Under these conditions, the mass of a moving body is independent of its speed. In the theory of relativity, attributed to Albert Einstein, it is not so. It is assumed that the current mass m depends on the rest mass m_0 by the formula

$$m = \frac{m_0}{\sqrt{1 - v^2 / c^2}} , \qquad \text{(D1_1)}$$

where v is the current velocity of a moving body and c is the speed of light.

It is known that the velocity of the Earth, when it moves along its elliptic orbit, is approximately 30 km/s. In this particular case, the initial rest mass of 1 kg will change to 1.000000005000000 kg. Thus, for most of the earthbound tasks, we could safely accept the statement that the mass is of a body is independent of its velocity.

D1.1. Newtonian mechanics

Dynamics is focused on the determination of the motion of bodies with respect to forces and moments that are applied to them. Generally, the problems in dynamics lead to ordinary differential solutions requiring solving them in order to find displacements, velocities, and accelerations as functions of time. Consequently, the forces and moments are also functions of time. Recall, that in statics the problems led to solving the system of algebraic equations.

Newton described force as the ability to cause a body to accelerate. His three laws can be summarized as follows.

First law: If there is no net force on a body, then its velocity is constant. The body is either in rest (if its velocity is equal to zero), or it moves at constant speed in a straight line[1].

Second law: The time rate of momentum[2] , i.e, $\vec{p} = m\vec{v}$, of a particle is equal to the acting force $\vec{F}$, i.e., $\mathrm{d}\vec{p}/\mathrm{d}t = \vec{F}$.

Third law: When a first body exerts a force $\vec{F}_1$ on a second body, the second body simultaneously exerts a force $\vec{F}_2 = -\vec{F}_1$ on the first body. This means that forces $\vec{F}_1$ and $\vec{F}_2$ are equal in magnitudes and opposite in directions.

In this simple formulation, Newton's laws of motion are valid only in inertial frames of reference. That is in frames that are not subjected to acceleration or by other words in frames that are either stationary or move (without rotation) with a constant velocity.

Newton's second law, written for a particle of mass *m*, states that the time rate of momentum is proportional to the external force

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \frac{\mathrm{d}(m\vec{v})}{\mathrm{d}t} = \vec{F} \Rightarrow \frac{\mathrm{d}m}{\mathrm{d}t}\vec{v} + \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}m = \vec{F}. \qquad \text{(D1\_2)}$$

If the mass of the particle does not change in time, i.e. $m = konst$, then the most frequently used formulation of Newton's law is

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}m = \vec{F} \Rightarrow m\vec{a} = \vec{F}. \qquad \text{(D1\_3)}$$

Another available formulation

$$\mathrm{d}(m\vec{v}) = \vec{F}\mathrm{d}t, \qquad \text{(D1\_4)}$$

stating that the time rate of momentum is equal to the impulse of external force, is convenient for cases when the mass quantity depends on time. A starting rocket, consuming its fuel at high rates, is a good example.

Example – time rate of momentum equals the pulse of external force

Given: A loose freight car of mass m has initial velocity v_0. Assume that the overall resistance effects are approximated by a force which is $mg/200$ and acts against the motion.
Determine: Time interval after which the car stops.

[1]This law was initially deduced by Galileo. Before him, in agreement with Aristotelian mechanics, it was firmly believed that objects that are not being pushed or pulled have a tendency to stop.
[2] By momentum, sometimes linear momentum, is understood the product of mass and velocity.

For a freight car considered as the particle in the straight line motion, we can write

$$m\mathrm{d}v = P\mathrm{d}t$$

or

$$mv_{\mathrm{k}} - mv_{\mathrm{z}} = \int_{t_{\mathrm{z}}}^{t_{\mathrm{k}}} P(t)\mathrm{d}t\,,$$

where

... $v_{\mathrm{k}}, v_{\mathrm{z}}$ are velocities at the end and at the beginning of the observed phenomenon,
... $t_{\mathrm{k}}, t_{\mathrm{z}}$ are times corresponding to the end and to the beginning of time interval.

Considering a force whose time distribution is constant we have

$$mv_{\mathrm{k}} - mv_{\mathrm{z}} = \int_{t_{\mathrm{z}}}^{t_{\mathrm{k}}} P(t)\mathrm{d}t = P\left(t_{\mathrm{k}} - t_{\mathrm{z}}\right) = P\Delta t\,, \qquad \text{(a)}$$

where

$\Delta t = \left(t_{\mathrm{k}} - t_{\mathrm{z}}\right)$ ... is the corresponding time interval.

Using Eq. (a) and assuming that the resistance force is constant we get

$$0 - mv_{\mathrm{z}} = -\frac{mg}{200}\Delta t \quad \Rightarrow \quad \Delta t = \frac{200v_{\mathrm{z}}}{\mathrm{g}}.$$

Discussion.
In this simplified case the time to stop is independent of the mass of the freight car. What are the limits of the accepted simplification?

D1.2. Important definitions to remember

Force – the cause of the change of motion.
Matter commonly exists in four *states* (or *phases*), i.e. solid, liquid, gas, and plasma. It has many properties as *volume, density*, *color*, *temperature*, *mass* and also the *weight*.

Mass – the measure of the unwillingness of matter to change its state of motion. It is independent of gravitational field. It is measured in [kg].

Weight – one of the matter properties. It depends on the gravitational field. It is measured in [N].

D1.3. SI metric system

of units is the standard that is commonly used in this textbook. In the SI system the quantities as mass, length and time are considered as basic mechanical units, in contradistinction to the old technical system of units in which force, length and time were taken as the basic ones.

SI metric system of units | Old technical system of units

Basic units

mass	[kg]	… kilogram	force	[kp]	… kilopond
length	[m]	… meter	length	[m]	… meter
time	[s]	… second	time	[s]	… second

Derived units

force	$[\mathrm{N}] = [\mathrm{kgm/s^2}]$ … newton	mass	$[\mathrm{kp\,s^2/m}]$ … has no name

D1.4. Work, energy, power and corresponding units

D1.4.1. Work

Work = **force** × **displacement**. This simple statement is valid only if both vector components are constant and have the same line of action. Otherwise, an incremental approach has to be used.

The increment of work is $\mathrm{d}L = \vec{F} \cdot \mathrm{d}\vec{s} = F\,\mathrm{d}s\,\cos\varphi$, where φ is the angle between $\vec{F}$ and $\vec{s}$.

Example – increment of work

Given: A perfectly flexible rope of the length l hangs vertically in the gravitational field. The 'longitudinal density', that is the mass of one meter of the rope, is γ [kg/m].
Determine: The work needed to wind up the full length of the rope at its upper end.

The mass of the rope element $\mathrm{d}x$, whose distance from the its upper end is x, is $dm = \gamma\,\mathrm{d}x$. The elementary work needed for its raising by x is

$\mathrm{d}L = xg\,\mathrm{d}m = \gamma\,g\,x\,\mathrm{d}x$, where g is gravitational acceleration.

The cumulative effort for the task is obtained by integration

$L = \gamma\,g\int_0^l x\,\mathrm{d}x = \frac{1}{2}\gamma\,gl^2$. Dimensional check: $\frac{\mathrm{kg}}{\mathrm{m}}\frac{\mathrm{m}}{\mathrm{s}^2}\mathrm{m}^2 = \mathrm{kg}\frac{\mathrm{m}}{\mathrm{s}^2}\mathrm{m} = \mathrm{Nm} = \mathrm{J}$. Stimmt.

Note: The weight of the rope of the length l is $\gamma\,g\,l$. The work needed to raise the whole rope by the distance l, without the winding, is $\gamma\,gl^2$. By the above reasoning, we came to one half of it, only. Why? This is due to the fact that by subsequent winding a shorter and shorter length of the rope is being raised.

D1.4.2. Energy

SI system	Old technical system	
$\mathrm{J} = \mathrm{Nm}$, Joule = Newton × meter	kpm,	kp × meter
$1\,\mathrm{J} = 0{,}102\ \mathrm{kpm}$	$1\,\mathrm{kpm} = 9{,}81\ \mathrm{J}$	
Recall also	$1\,\mathrm{kpm} = 2{,}343\ \mathrm{cal}$,	$1\,\mathrm{kcal} = 427\,\mathrm{kpm}$

D1.4.3. Power

is the rate of work, i.e. work exerted per unit of time. It is measured in SI watts $[\mathrm{W}]$ or in old-fashioned horsepowers [hp]. One has to distinguish the metric horsepower, denoted $[\mathrm{hp} = \mathrm{hp}_{\mathrm{metric}}]$ and the British or imperial horsepower $[\mathrm{hp}_{\mathrm{imperial}}]$, respectively.

$\mathrm{W} = \mathrm{J/s}$	$1\,\mathrm{hp} = 1\,\mathrm{hp}_{\mathrm{metric}} = 75\,\mathrm{kpm/s}$
$1\,\mathrm{kW} = 0{,}736\ \mathrm{hp}$	$1\,\mathrm{hp}_{\mathrm{metric}} = 735{,}5\,\mathrm{W},\ 1\,\mathrm{hp}_{\mathrm{imperial}} = 745{,}7\,\mathrm{W}\ \ldots)$[3]
$\mathrm{Ws} = \mathrm{J}$	
$1\,\mathrm{kWh} = 3{,}6\times10^{6}\ \mathrm{J} = 367\,000\ \mathrm{kpm}$	

D1.4.4. Potential and kinetic energies

Generally, we say that the energy is an ability to work.

If the body of mass m, in the Earth gravitational field, is raised (or lifted) to the height h, then the work required (or the work done) is

$$W = E_{\mathrm{p}} = mgh\,.$$

This way, the body being raised gains the potential energy E_{p}.

If the body is released (with zero initial velocity) from the elevated position, defined by the height h, it hits the initial position (ground) by velocity v, which might be determined from the following equation of motion

$$ma = mg\,,$$

$$\frac{\mathrm{d}v^2}{2\mathrm{d}x} = g\,,$$

$$\int_0^v \mathrm{d}v^2 = 2g\int_0^h dx\,,$$

$$v^2 = 2gh \Rightarrow h = \frac{v^2}{2g}\,.$$

[3] For a detailed explanation see https://en.wikipedia.org/wiki/Horsepower#British_horsepower.

Work exerted (gained) by the falling body from the height h is also mgh, so

$W = mgh$.

Substituting $h = \frac{v^2}{2g}$ into the previous equation we get the kinetic energy in the form

$E_\mathrm{k} = mgh = \frac{1}{2}mv^2$.

The sum of potential and kinetic energy, at any moment, is constant.

For the rate of kinetic energy (for a mass particle), using the vector notation, we can write

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \sum \mathbf{F}_i ,$$

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\mathrm{d}\mathbf{r} = \sum \mathbf{F}_i \,\mathrm{d}\mathbf{r} \text{, but } \mathrm{d}\mathbf{r} = \mathbf{v}\,\mathrm{d}t \text{, so}$$

$$m\,\mathbf{v}\mathrm{d}\mathbf{v} = \sum \mathbf{F}_i \,\mathrm{d}\mathbf{r} ,$$

$$m\int_{\mathbf{v}_0}^{\mathbf{v}} \mathbf{v}\mathrm{d}\mathbf{v} = \int \sum \mathbf{F}_i \,\mathrm{d}\mathbf{r} ,$$

$$\frac{1}{2}m\left(\mathbf{v}^2 - \mathrm{v}_0^2\right) = W ,$$

$$E_\mathrm{k} - E_{\mathrm{k}0} = W .$$

The change of kinetic energy is equal to the work done by exerting (applied) forces.

Since the work is defined as power$\times$time, we might use the formula $W = P\Delta t$ [4]. Similarly, the power could be defined as the time rate of energy, so

$$dE_\mathrm{k} = P\,\mathrm{d}t \Rightarrow \frac{dE_\mathrm{k}}{\mathrm{d}t} = P .$$

The time rate of kinetic energy is equal to the power of applied forces.

Example – difference of kinetic energies equals the work done by exerting forces

Given: A particle in the gravitational field of the Earth, having mass m, is released with zero initial velocity from the height h. After a free fall of the vertical distance H the particle hits a non-linear spring which resists the consequent motion of the particle by a force $S = ky^3$, where k is the spring stiffness and y is the immediate deflection measured downwards from the undeformed length of the spring. It is assumed that the spring is massless.
Determine: The maximum deflection of the spring, say y_max, due to the motion of the particle together with the spring.

[4] The formula is valid the power P is constant during the time interval Δt.

The difference of the final kinetic energy (which, at the moment of the maximum spring deflection, is zero) and the initial kinetic energy (which is zero due to the zero initial velocity as well) is equal to the work exerted by external forces, i.e. by work done by the spring force and by the force of gravity, so

$$0-0=\int_0^{H+y_{\max}} mg\,\mathrm{d}y-\int_0^{y_{\max}} ky^3\mathrm{d}y\,,\ 0=mg\left(H+y_{\max}\right)-\frac{1}{4}ky_{\max}^4 \quad \Rightarrow y_{\max}\,.$$

Example – difference of kinetic energies equals the work of exerting forces

Given: To a crane truck is attached a vertically positioned rope of the length l that could freely swing. At the end of the rope there is a particle of mass m. Assume, that the truck suddenly stops.
Determine: After the truck stops, find the maximum horizontal distance x to which the load is displaced.

From Fig. D01 one sees the geometrical relation between x and h coordinates.

$$x=\sqrt{l^2-(l-h)^2}=\sqrt{h(2l-h)}$$

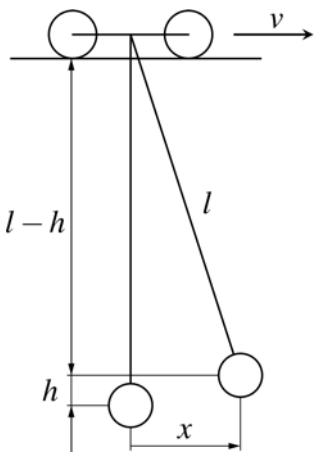


Fig. D01. Crane truck suddenly stops

After the crane truck is stopped the initial kinetic energy of the particle $\frac{1}{2}mv^2$ is transferred into its potential energy mgh. So,

$$\frac{1}{2}mv^2=mgh \quad \Rightarrow \quad h=\frac{v^2}{2g}\,.$$

Rearranging and substituting we get

$$x=\sqrt{\frac{v^2}{2g}\left(2l-\frac{v^2}{2g}\right)}\,.$$

Example – energy conservation

Given: A circular pulley (radius r, mass m_2, moment of inertia with respect to its centre J_S) turns with constant angular velocity ω about the axis passing through the joint S. A massless rope is wound around the pulley. At the end of rope, there is attached a load of mass m_1. See Fig. D02. At the beginning of the observed situation the driving torque applied to the pulley is suddenly stopped. Due to inertia the load m_1 for a moment still goes up before it stops.

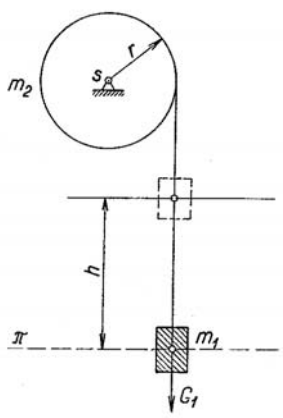


Fig. D02. The weight moves upwards due to inertia

Determine: The necessary initial angular velocity ω needed for the load to continue in its upward motion and to stop at the distance h. We have learned that in the absence of resistance the change of kinetic energy equals the change of potential energy, so

$$\Delta K = \Delta W ,$$

or

$K_\mathrm{b} - K_\mathrm{e} = W_\mathrm{b} - W_\mathrm{e}$ ... indices b and e indicate the beginning and the end of the observed situation.

In our case

$$K_b = \frac{1}{2} J_S \omega_z^2 + \frac{1}{2} m_1 v_z^2 , \qquad K_\mathrm{e} = 0 ,$$
$$W_\mathrm{b} = 0 , \qquad W_\mathrm{e} = mgh .$$

The kinematic relations are

$$v_\mathrm{z} = r\omega_\mathrm{z} .$$

Substituting we get

$\frac{1}{2} J_S \omega_z^2 + \frac{1}{2} m_1 r^2 \omega_z^2 - 0 = 0 - (-mgh)$ ... the minus sign indicates that the work is consumed,

$$\Rightarrow \omega .$$

D1.4.5. Potential forces

By potential forces are understood the forces whose directions and magnitudes depend on their positions only. As examples, the gravitational forces or the spring forces could be mentioned.

Example – exerted work does not depend on the trajectory

A particle of mass m with initial velocity v_0 slides along the frictionless trajectory depicted in Fig. D03.

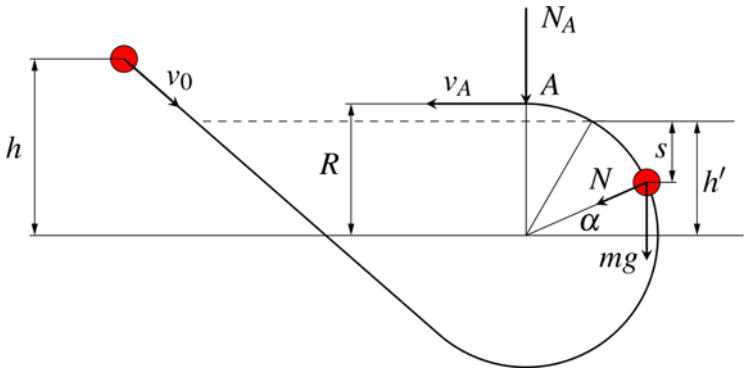


Fig. D03. Motion in gravitational field

Given: $m, v_0, R, h', f = 0$.
Determine:

a) The height h needed for the normal reaction between the particle and the circular part of the trajectory to have the reaction magnitude at the point A equal to $N_\mathrm{A} = \frac{1}{2}mg$.
b) The position s, where the particle loses it contact with the circular part of the trajectory if the particle is released from the height $h' < R$.

Add a)
The equation of motion written at A in the normal direction is

$$ma_\mathrm{n} = N_\mathrm{A} + mg. \tag{a}$$

The normal acceleration at the point A is

$$a_\mathrm{n} = \frac{v_\mathrm{A}^2}{\mathrm{R}}. \tag{b}$$

The required 'half-value' condition is

$$N_\mathrm{A} = \frac{1}{2}mg. \tag{c}$$

So,

$$ma_\mathrm{n} = \frac{1}{2}mg + mg = \frac{3}{2}mg \qquad \Rightarrow a_\mathrm{n} = \frac{3}{2}g. \tag{d}$$

The velocity satisfying the condition (c) is obtained by comparing Eqs. (b) a (d), thus

$$v_\mathrm{A}^2 = \frac{3}{2}Rg. \tag{e}$$

The difference of kinetic energies is equal to the work exerted by external forces,

$$\frac{1}{2}mv_\mathrm{A}^2 - \frac{1}{2}mv_0^2 = \int P\,\mathrm{d}s.$$

No resistance forces are considered, so the only working force is the force of gravity i.e. the weight. Since the gravitational force is of potential nature, then the work done by the weight force does not depend on the trajectory but on a difference of potential levels only. So

$$\frac{1}{2}mv_\mathrm{A}^2 - \frac{1}{2}mv_0^2 = mg(h - R). \tag{f}$$

It is obvious that $mg(h - R) > 0 \Rightarrow h > R$ and that the result does not depend on the mass of the considered body.

The unknown height h is obtained from Eq. (f), so

$$h = \frac{v_{\mathrm{A}}^2 - v_0^2}{2g} + R .$$

Add b)
In a generic position, denoted by so far unknown angle α, we can write the equation of motion for the normal direction in the form

$$ma_n = N + mg\sin\alpha .$$

The normal acceleration is given by $a_{\mathrm{n}} = \frac{v_{\mathrm{A}}^2}{R}$.

The loss of contact is given by the condition of zero contact force, thus

$$N = 0 .$$

From it follows that the release velocity has to be

$$\frac{v^2}{R} = g\sin\alpha \quad \Rightarrow \quad v^2 = Rg\sin\alpha .$$

Again, the difference of kinetic energies is equal to the work exerted by external forces between two potential levels. that is $s = h' - R\sin\alpha$. Thus

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = mgs ,$$
$$\frac{1}{2}Rg\sin\alpha - \frac{1}{2}v_0^2 = g\left[h' - R\sin\alpha\right],$$
$$\sin\alpha = \frac{v_0^2 + 2gh'}{3Rg} .$$

The condition of the existence of the release point within the first quadrant, i.e. $0 \le \alpha \le \pi/2$, requires

$$0 \le \frac{v_0^2 + 2gh'}{3Rg} \le 1 \quad \Rightarrow \quad 0 \le v_0 \le \sqrt{g(3R - 2h'} .$$

Alternatively, if $R = h'$, then for $\alpha = \pi/2$ the initial velocity has to be $v_0 = \sqrt{2gh'}$.

D1.4.6. Momentum, sometimes linear momentum

Momentum for a particle is the product of mass and its velocity, i.e. $\vec{p} = m\vec{v}$.

D1.4.7. Angular momentum, sometimes moment of momentum or rotational momentum

Angular momentum for a particle is the cross product $\vec{r} \times \vec{p}$, where $\vec{r}$ is the particle's position vector relative to a specified origin.

D1.4.8. Velocity and speed

These terms are often distinguished. In this text it is understood that velocity is a vector and that the speed is just its magnitude. In this sense, we might write $v = |\vec{v}|$.

D2. Dynamics of a particle subjected to a straight line motion

In this case, one can write

$$\sum_{i=1}^{n} \vec{P}_i + \sum_{j=1}^{m} \vec{R}_j = m\vec{a}\,, \qquad \text{(D2\_1)}$$

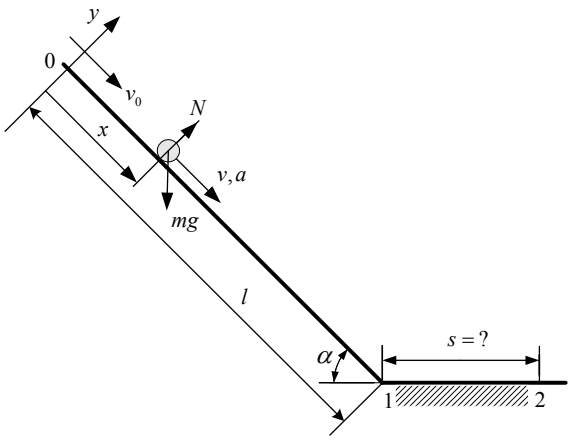
meaning that external forces plus reaction forces are equal to the inertial force.

If the motion is assumed in the direction of the x coordinate axis only, then the scalar notation yields

$$\begin{aligned} x:&\quad \sum P_{xi} + \sum R_{xj} = ma_x, \\ y:&\quad \sum P_{yi} + \sum R_{yj} = 0, \\ z:&\quad \sum P_{zi} + \sum R_{zj} = 0. \end{aligned} \qquad \text{(D2\_2)}$$

Example – particle on an inclined plane

Given: A particle of mass m, having the initial velocity v_0, moves downward an inclined plane defined by the angle α and the length l. See Fig. D04. When the particle reaches the point 1, then continues to move along the straight horizontal line. Friction phenomena are characterized by the coefficient of friction f.
Determine: The distance s, indicated by the point 2, where the particle stops.



Fig_D04. The particle on a slope

It is convenient to choose a suitably defined coordinate system – at first, the positive direction of x axis should be introduced in the assumed direction of the particle motion. Also, the analyzed particle should be considered at a generic position, defined by the indicated x distance – not at the beginning, nor at the end. It is also a good habit to set the unknown positive direction of acceleration in the directions of positive coordinate axes.

As in statics, a free body diagram approach is used. In this case, the external forces acting on the particle consist of the normal force, the gravity force and the friction force. And according to Newton's law, the vector sum of these forces has to be equal to the vector of inertia force. Instead of equilibrium equations, as in statics, we write the equations of motion. Their scalar form, for the situation between points 1 and 2, is

$$x: mg\sin\alpha - Nf = ma\,,$$
$$y: N - mg\cos\alpha = 0\,.$$

Extracting the normal force from the second equation and substituting it into the first one we get

$$mg(\sin\alpha - f\cos\alpha) = ma\,.$$

Observing this equation of motion we deduce that the solution is independent of the mass of the particle since the equation can be reduced by a factor of m. Thus, the quantity a is constant, so the particle moves with a constant acceleration. But for the assumed downward motion this acceleration has to be positive, so the condition of the task solvability is

$$(\sin\alpha - f\cos\alpha) > 0, \quad \Rightarrow f < \frac{\sin\alpha}{\cos\alpha} = \tan\alpha\,.$$

Known kinematic relation, i.e. $a = \dfrac{v\,\mathrm{d}v}{\mathrm{d}x}$, allows expressing the equation of motion in term of velocity, i.e.

$$g(\sin\alpha - f\cos\alpha) = \frac{v\,\mathrm{d}v}{\mathrm{d}x}\,.$$

Integrating the last equation, between points 0 and 1, gives

$$g(\sin\alpha - f\cos\alpha)\int_0^l \mathrm{d}x = \int_{v_0}^{v_1} v\,\mathrm{d}v\,,$$
$$gl(\sin\alpha - f\cos\alpha) = \frac{v_1^2 - v_0^2}{2}\,.$$

So, the velocity at the end of the inclined plane, i.e. at the point 1, is

$$v_1 = \sqrt{2gl(\sin\alpha - f\cos\alpha) + v_0^2}\,.$$

What happens next? The final velocity of the first part of the motion becomes a starting velocity for the second part of the motion. Denoting new kinematic quantities and the new coordinate system by primes, we have a new equation of motion, which is valid between points 1 and 2. It has the form

$$-mgf = ma'\,,$$
$$-gf = \frac{v'\mathrm{d}v'}{\mathrm{d}x'}. \qquad \text{(a)}$$

Initial conditions are

$t = 0$ $x' = 0$ and $\dot{x}' = v_1$.

When the particle stops, its final velocity is equal to zero. So, integrating Eq. (a) within the proper limits gives

$$-gf\int_0^s dx' = \int_{v_1}^0 v'\mathrm{d}v'$$

and we get the answer in the form

$$-gfs = -\frac{v_1^2}{2} \Rightarrow s = \frac{v_1^2}{2gf},$$

where the velocity at point 1 was found earlier as

$$v_1 = \sqrt{2gl(\sin\alpha - f\cos\alpha) + v_0^2}.$$

D3. Dynamics of a particle subjected to a motion along a curve

is explained using a simple example.

Example – oblique throw in plane

Given: A projectile, considered as a particle of mass m, is shot (thrown) from the origin of the coordinate system with initial velocity v_0 under an angle α. The air resistance is neglected. The 'terrain' is idealized by a straight line originating at the origin and defined by the angle β. It is assumed that $\alpha > \beta$. See Fig. D05.
Determine: The trajectory of the projectile, the hit point coordinates, and the hit velocity.

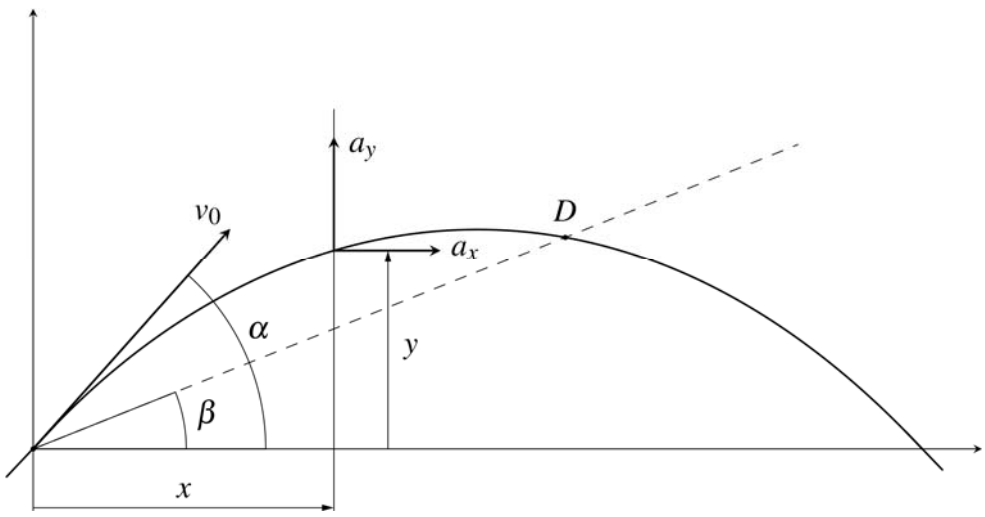


Fig. D05. Oblique shot

Again, we start with a particle being considered at a generic position defined by x and y coordinates. The positive directions of acceleration components are assumed in positive directions of coordinate axes. The only external force, the weight of the particle, points downwards. A particle in the plane has two degrees of freedom. Thus, two scalar equations of motion are written in x and y directions respectively, and consequently integrated within the proper limits, i.e. from the beginning ($t = 0$) to the current position at the time t, characterized by x and y coordinates and by generic values of velocities and accelerations.

$$ma_x = 0, \qquad ma_y = -mg,$$

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}v_y}{\mathrm{d}t} = -g,$$

$$\int_{v_0 \cos\alpha}^{v_x} \mathrm{d}v_x = 0, \qquad \int_{v_0 \sin\alpha}^{v_y} \mathrm{d}v_y = -g\int_0^t \mathrm{d}t,$$

$$v_x = v_0 \cos\alpha, \qquad v_y = v_0 \sin\alpha - gt.$$

The above result shows that the x component of velocity is constant, while the y component is a linear function of time. Another integration gives the parametric components of the trajectory in the form

$$x = v_0 t \cos\alpha, \quad y = v_0 t \sin\alpha - \frac{1}{2} g t^2 .$$

To get the trajectory of the motion in another form, i.e. $y = f(x)$, we have to extract time t from the first equation

$$t = \frac{x}{v_0 \cos\alpha} \tag{a}$$

and substitute it into the second equation. Thus, the trajectory of the particle is described by a parabolic function

$$y = \frac{v_0 \sin\alpha}{v_0 \cos\alpha} x - \frac{g}{2 v_0^2 \cos^2\alpha} x^2 = x \tan\alpha - \frac{g}{2 v_0^2 \cos^2\alpha} x^2 . \tag{b}$$

We have defined the 'terrain' by the equation

$$y = x \tan(\beta) \ . \tag{c}$$

The hit point coordinates, say $(x_\mathrm{D}, y_\mathrm{D})$, can be obtained by equaling y coordinates in Eqs. (b) and (c). So,

$$x_\mathrm{D} \tan\beta = x_\mathrm{D} \tan\alpha - \frac{g}{2 v_0^2 \cos^2\alpha} x_\mathrm{D}^2 ,$$

$$\tan\beta - \tan\alpha = -\frac{g}{2 v_0^2 \cos^2\alpha} x_\mathrm{D} \ ,$$

$$x_\mathrm{D} = \frac{2(\tan\alpha - \tan\beta) v_0^2 \cos^2\alpha}{g} \qquad \text{and consequently} \qquad y_\mathrm{D} = x_\mathrm{D} \tan\beta .$$

The time to hit, obtained from Eq. (a), is

$$t_\mathrm{D} = \frac{x_D}{v_0 \cos\alpha} .$$

The components and the magnitude of the hit velocity are

$$v_{\mathrm{D}x} = v_0 \cos\alpha ,$$

$$v_{\mathrm{D}y} = v_0 \sin\alpha - g t_\mathrm{D} ,$$

$$v_\mathrm{D} = \sqrt{v_{Dx}^2 + v_{\mathrm{D}y}^2} \ .$$

Check that the magnitude of the hit velocity, in the absence of the air resistance, is the same as that of the initial velocity.

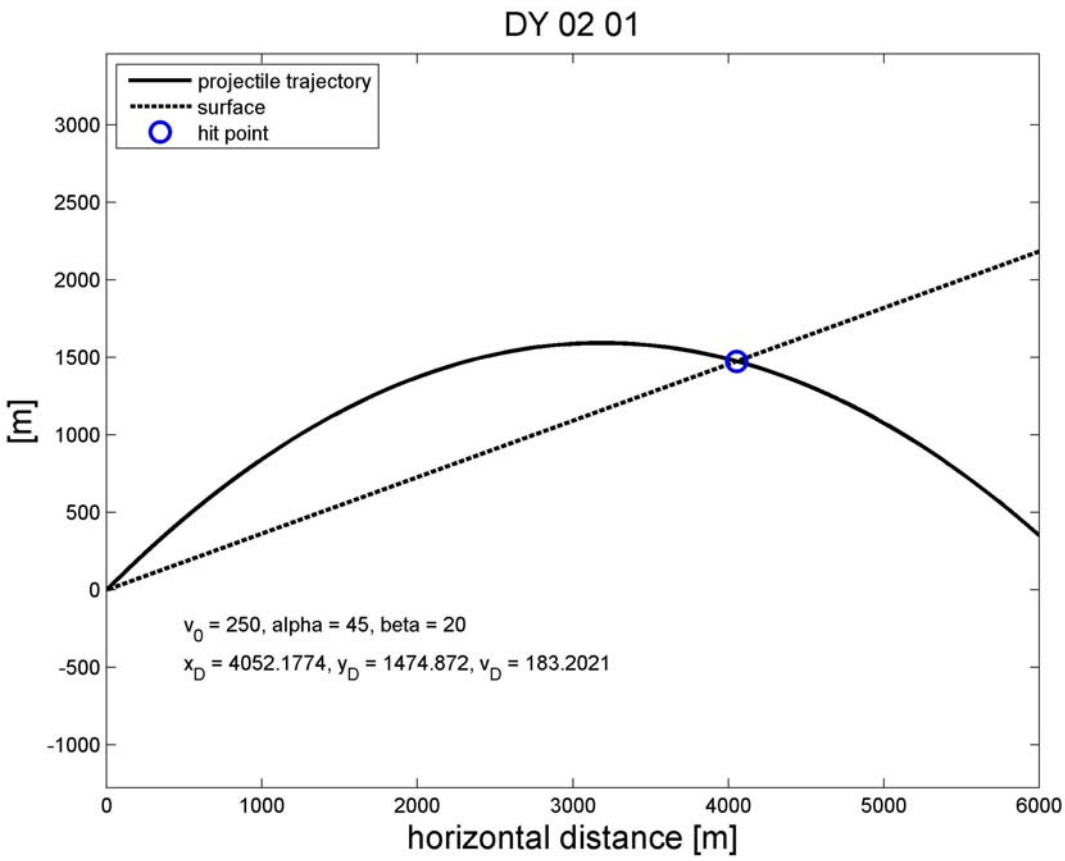


Fig. D06. Hit point coordinates

See the program `D01_oblique_throw.m` and Fig. D06.

```
% D01_oblique_throw
% original file name is edu_UL_2013_DY_02_01_sikmy_vrh
clear
m = 5; v0 = 250; alf = 45; bet = 20; g = 9.81;
alfa = alf*pi/180; beta = bet*pi/180;
x_range = 0:1:6000;
% projectile trajectory
yp = x_range*tan(alfa) - x_range.*x_range*g/(2*v0^2*cos(alfa)^2);
% surface
ys = x_range*tan(beta');
% hit point
xD = 2*(tan(alfa) - tan(beta))*v0^2*cos(alfa)^2/g;
yD = xD*tan(beta);
% hit time
tD = xD/(v0*cos(alfa));
% hit velocity
vDx = v0*cos(alfa);
vDy = v0*sin(alfa) - g*tD;
vD = sqrt(vDx^2 + vDy^2);
figure(1)
plot(x_range,yp,'k-', x_range,ys,'k--', xD,yD,'o', 'linewidth', 2, 'markersize', 10);
axis('square'); axis('equal')
title('DY 002 01', 'fontsize', 16)
txt1 = ['v_0 = ' num2str(v0) ', alpha = ' int2str(alf) ', beta = ' int2str(bet)];
txt2 = ['x_D = ' num2str(xD), ', y_D = ' num2str(yD) ', v_D = ' num2str(vD)];
legend('projectile trajectory', 'surface', 'hit point', 2)
text(500,-250, txt1)
text(500,-500, txt2)
xlabel('horizontal distance [m]', 'fontsize', 16); ylabel('[m]', 'fontsize', 16)
print -djpeg -r300 fig_DY_02_01_02
```

Example – long jump record, a theoretical limit

Determine: The theoretical maximum length of a long jump record, assuming that the contestant, immediately before the recoil, has the velocity, corresponding to the average velocity of Usain Bolt during his world record on 100 m. Assume the recoil angle of 20 degrees. The jumper is considered as a mass particle and the air resistance is neglected. The world record holder on 100 m (2014) is Usain Bolt and his time is 9.58 s.

Compare the computed long jump result with the present (2014) record which, is 8.95 m and is attributed to Mike Powell. Consider other recoil angles for comparison.

Hint: Use the relations derived in the previous example, where the parametric equations of the trajectory were obtained in the form

$$x = v_0 t \cos\alpha, \quad y = v_0 t \sin\alpha - \frac{1}{2} g t^2 .$$

Eliminating the time variable from the first equation

$$t = \frac{x}{v_0 \cos\alpha} \tag{a}$$

and substituting it into the second equation, we get an alternative form $y = f(x)$ in the form

$$y = \frac{v_0 \sin\alpha}{v_0 \cos\alpha} x - \frac{g}{2 v_0^2 \cos^2\alpha} x^2 = x \tan\alpha - \frac{g}{2 v_0^2 \cos^2\alpha} x^2 . \tag{b}$$

Equation of the surface was defined by $y = x \tan(\beta)$. (c)

Now, the 'terrain' is a straight line again, but defined by $\beta = 0$.

Impact point, with coordinates (x_D, y_D), is obtained by equaling *y*-coordinates in equations (b) and (c), respectively. Thus

$$x_D \tan\beta = x_D \tan\alpha - \frac{g}{2 v_0^2 \cos^2\alpha} x_D^2 ,$$

$$\tan\beta - \tan\alpha = -\frac{g}{2 v_0^2 \cos^2\alpha} x_D .$$

Impact coordinates are

$$x_D = \frac{2(\tan\alpha - \tan\beta) v_0^2 \cos^2\alpha}{g}, \; y_D = x_D \tan\beta, \text{ but in this case } \beta = 0 .$$

See the program `D02_long_jump.m`, which we obtain as a slight modification of `D01_oblique throw.m`. The program solves the task for a single initial velocity, but for three different recoil angles, i.e. 20, 20.844 and 45 degrees respectively. The respective long jump

‘records’ are 7.12, 8.95 a 11.11 m respectively. The middle recoil value was chosen in such a way that it corresponds to Powell’s world record. The results are in Fig. D07.

Analysis

The length of the jump is generally longer for a greater initial velocity and a greater recoil angle as well. The maximum theoretical value for any initial velocity is obtained for the recoil angle of 45 degrees. In this case, the jumper would have to reach the height of almost three meters. It is obvious that such a value is physiologically unattainable.

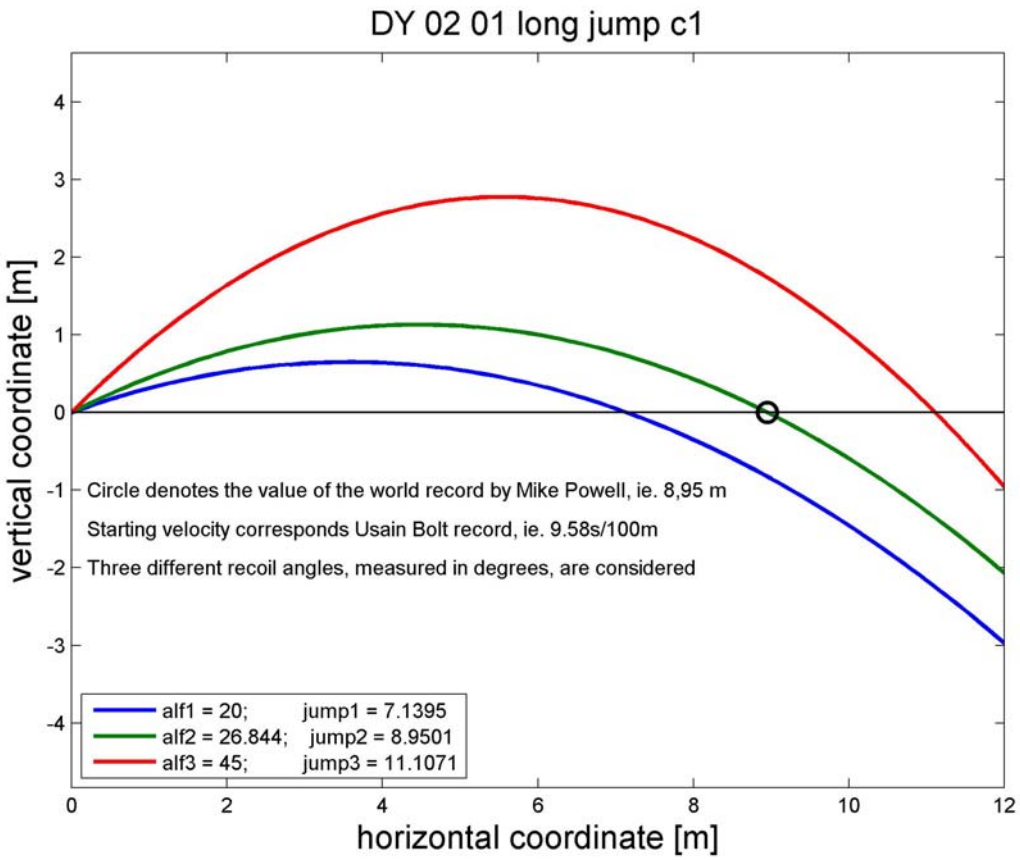


Fig. D07. Long jump limit

Program `D02_long_jump_c1.m`

```
% D02_long_jump_c1.m
% original file name is edu_UL_2013_DY_02_01_long_jump_c1
% find the theoretical value of the long jump
% assume that the initial speed is given by Usain Bolt record for 100 m
% three values of recoil angle are consiered
clear
time_Usain_Bolt = 9.58;                     % [s/100 m] ... time for the world record
velocity_U_B = 100/time_Usain_Bolt;         % ... velocity in m/s
v0 =  velocity_U_B;
m = 5; alf = 25; bet = 0; g = 9.81; beta = bet*pi/180;
alf_all = [20 26.844 45];                % try three different recoil angles
alfa_range = alf_all*pi/180;
x_range = 0:0.1:12;
% projectile
i = 0
for alfa = alfa_range
    i = i + 1;
    % trajectory
    yp(:,i) = x_range*tan(alfa) - x_range.*x_range*g/(2*v0^2*cos(alfa)^2);
```

```
% surface idealized by a straight line
    ys(:,i) = x_range*tan(beta');
% hit point
    xD(i) = 2*(tan(alfa) - tan(beta))*v0^2*cos(alfa)^2/g;
    yD(i) = xD(i)*tan(beta);
% hit time
    tD(i) = xD(i)/(v0*cos(alfa));
% hit velocity
    vDx(i) = v0*cos(alfa);
    vDy(i) = v0*sin(alfa) - g*tD(i);
    vD(i) = sqrt(vDx(i)^2 + vDy(i)^2);
end
%
xD_Mike_Powell = 8.95; % long jump world record in [m]
yD = 0;
%
figure(1)

plot(x_range,yp, 'linewidth', 2);
hold on
plot(x_range,ys,'k-','linewidth', 1);
axis('square'); axis('equal')
title('DY 02 01 long jump c1', 'fontsize', 16)
txt1 = ['Circle denotes the value of the world record by Mike Powell, ie. 8,95 m'];
txt2 = ['Starting velocity corresponds Usain Bolt record, ie. 9.58s/100m'];
txt3 = ['Three different recoil angles, measured in degrees, are considered'];
lab1 = ['alf1 = ' num2str(alf_all(1)) ';          jump1 = ' num2str(xD(1))];
lab2 = ['alf2 = ' num2str(alf_all(2)) ';    jump2 = ' num2str(xD(2))];
lab3 = ['alf3 = ' num2str(alf_all(3)) ';          jump3 = ' num2str(xD(3))];
legend(lab1, lab2, lab3, 3)
text(0.2,-1, txt1)
text(0.2,-1.5, txt2)
text(0.2,-2, txt3)

plot(xD_Mike_Powell, yD, 'ok', 'linewidth', 2, 'markersize', 10)
hold off

xlabel('horizontal coordinate [m]', 'fontsize', 16);
ylabel('vertical coordinate [m]', 'fontsize', 16)
print -djpeg -r300 fig_DY_02_01_02_long_jump_c1
```

Example – moon landing

Given: α, β, v_0, H, m. A moon probe of the mass m is presently at the height H above the Moon surface. The moon gravity is considered as $g_m = g/6$. See Fig. D08.

Determine: The magnitude of the braking force F, sufficient for decreasing the initial velocity v_0, in such a way that the vertical direction of the landing velocity is zero.

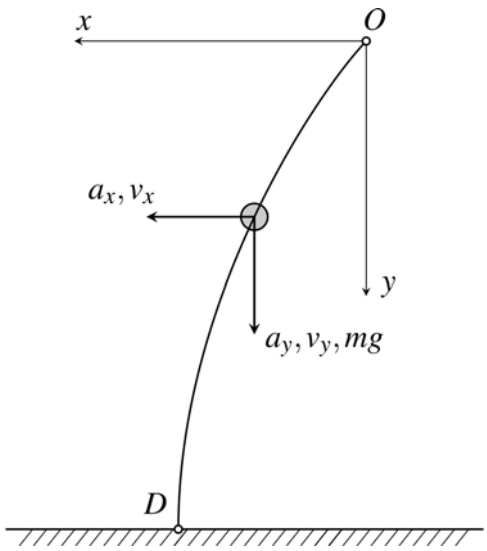


Fig. D08. Moon landing

Equations of motion and their consecutive integrations give

$$ma_x = -F\sin\beta\,, \qquad ma_y = \frac{1}{6}mg - F\cos\beta\,,$$

$$a_x = -\frac{F}{m}\sin\beta\,, \qquad a_y = \frac{g}{6} - \frac{F}{m}\cos\beta\,,$$

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = -\frac{F}{m}\sin\beta\,, \qquad \frac{dv_y}{dt} = \frac{g}{6} - \frac{F}{m}\cos\beta\,,$$

$$\int_{v_0\sin\alpha}^{v_x}\mathrm{d}v_x = -\frac{F}{m}\sin\beta\int_0^t \mathrm{d}t\,, \qquad \int_{v_0\cos\alpha}^{v_y}\mathrm{d}v_y = \left(\frac{g}{6} - \frac{F}{m}\cos\beta\right)\int_0^t \mathrm{d}t\,,$$

$$v_x = v_0\sin\alpha - \frac{Ft}{m}\sin\beta\,, \qquad \text{(a)} \qquad v_y = v_0\cos\alpha + \left(\frac{g}{6} - \frac{F}{m}\cos\beta\right)t\,, \qquad \text{(b)}$$

$$\int_0^{s_x}\mathrm{d}s_x = \int_0^t v_x\mathrm{d}x = \int_0^t\left(v_0\sin\alpha - \frac{Ft}{m}\sin\beta\right)\mathrm{d}t\,,$$

$$\int_0^{s_y}\mathrm{d}s_y = \int_0^t v_y\mathrm{d}t = \int_0^t\left[v_0\cos\alpha + \left(\frac{g}{6} - \frac{F}{m}\cos\beta\right)t\right]\mathrm{d}t\,,$$

$$s_x = v_0 t\sin\alpha - \frac{Ft^2}{2m}\sin\beta\,, \qquad \text{(c)}$$

$$s_y = v_0 t\cos\alpha + \frac{1}{2}\left(\frac{g}{6} - \frac{F}{m}\cos\beta\right)t^2\,. \qquad \text{(d)}$$

Simultaneous satisfaction of two conditions is required. After traveling the vertical distance H, the vertical component of the landing velocity should be zero. Using Eqs. (b) and (d) we get

$$0 = v_0\cos\alpha + \left(\frac{g}{6} - \frac{F}{m}\cos\beta\right)t\,, \qquad \text{(e)}$$

$$H = v_0 t\cos\alpha + \frac{1}{2}\left(\frac{g}{6} - \frac{F}{m}\cos\beta\right)t^2\,. \qquad \text{(f)}$$

These two equations allow to determine the time to landing t_D and the magnitude of the braking force F. Excluding time from Eq. (e)

$t = \dfrac{v_0\cos\alpha}{\dfrac{g}{6} - \dfrac{F}{m}\cos\beta}$ and substituting it into Eq. (f) we get

$$H = \frac{v_0^2\cos^2\alpha}{\dfrac{F}{m}\cos\beta - \dfrac{g}{6}} + \frac{1}{2}\left(\frac{g}{6} - \frac{F}{m}\cos\beta\right)\frac{v_0^2\cos^2\beta}{\left(\dfrac{F}{m}\cos\beta - \dfrac{g}{6}\right)^2}\,.$$

This leads to a quadratic equation for the variable F in the form

$$aF^2 + bF + c = 0,$$

where

$$a = \frac{H\cos^2\beta}{m^2},$$

$$b = -\frac{2gH\cos\beta}{6m} - \frac{v_0^2\cos^2\alpha\cos\beta}{m} + \frac{v_0^2\cos^2\alpha\cos\beta}{2m},$$

$$c = \frac{g^2H}{36} + \frac{gv_0^2\cos^2\alpha}{6} - \frac{gv_0^2\cos^2\alpha}{12}.$$

From the relation $F = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ only a positive root is meaningful. You might check that the vertical component of the landing velocity is really equal to zero. But, a more detail analysis of the solution reveals that the horizontal component is generally non zero. So, the above conditions are not sufficient for a successful landing.

See the program `D03_moon_landing.m` and Fig. D09.

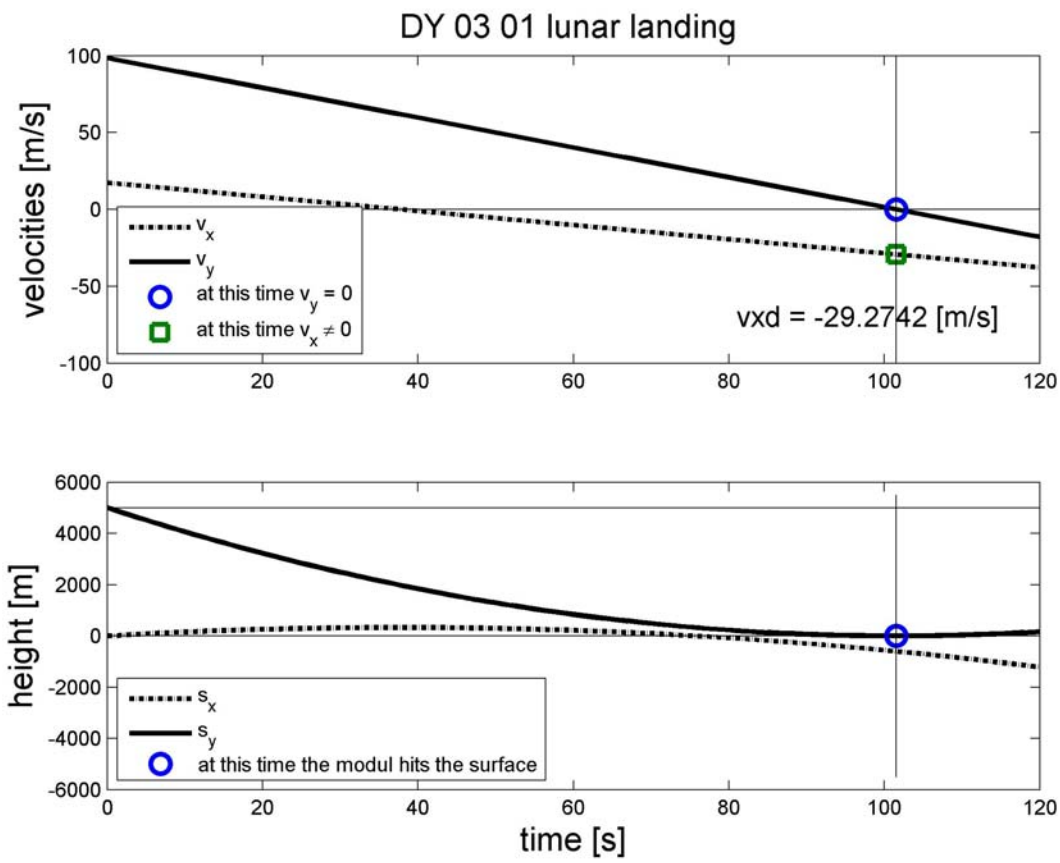


Fig. D09. Moon landing – the results we are not satisfied with

```
% D03_moon_landing
clear
H = 5000;                       % height in metres
alfa_d = 10; beta_d = 10;       % angles in degrees
alfa = pi*alfa_d/180; beta = pi*beta_d/180;
m = 1000;       % mass of lunar modul
g = 9.81;       % gravitional acceleration at earth
v0 = 100;       % initial modul velocity
tmax = 120;     % maximum time

% find the braking force
a = H*cos(beta)^2/m^2;
b = -2*g*H*cos(beta)/(6*m) - v0^2*cos(alfa)^2*cos(beta)/m + ...
    v0^2*cos(beta)*cos(alfa)^2/(2*m);
c = g^2*H/36 + g*v0^2*cos(alfa)^2/6 - g*v0^2*cos(alfa)^2/12;
FF1 = (-b + sqrt(b^2 - 4*a*c))/(2*a);
F = FF1;

incr_t = 1;
t = 0:incr_t:tmax;
len_t = length(t);
t_ones = ones(1,len_t);
vx = v0*sin(alfa)*t_ones - F*t*sin(beta)/m;
vy = v0*cos(alfa)*t_ones + (g/6 - F*cos(beta)/m)*t;
sx = v0*sin(alfa)*t - 0.5*F*t.^2*sin(beta)/m;
sy = v0*cos(alfa)*t + 0.5*(g/6 - F*cos(beta)/m)*t.^2;
td = v0*cos(alfa)/(F*cos(beta)/m - g/6)         % time of landing
F_krit = m*g/(6*cos(beta));
vx_d = v0*sin(alfa) - F*td*sin(beta)/m          % vx velocity at landing
sx_d = v0*sin(alfa)*td - 0.5*F*td^2*sin(beta)/m % x coor of landing

figure(1)
xx1 = [0 tmax]; yy1 = [H H];
xx2 = [td td]; yy2 = [-v0 v0]; yy3 = [-1.1*H 1.1*H];
yy4 = [0 0];
subplot(2,1,1)
plot(t,vx,'k-.', t,vy,'k-', td,0,'o', td,vx_d,'s', 'linewidth',2.5, 'markersize',10);
title('DY 03 01 lunar landing ', 'fontsize', 16)
ylabel('velocities [m/s]', 'fontsize', 16)
txt1 = ['vxd = ' num2str(vx_d) ' [m/s]']; text(81,-70,txt1, 'fontsize', 14)
hold on
plot(xx2, yy2, 'k', xx1,yy4, 'k')
hold off
legend('v_x', 'v_y', 'at this time v_y = 0', 'at this time v_x \neq 0', 3);% grid

subplot(2,1,2)
plot(t,sx,'k-.', t,H-sy,'k-', td,0,'o', 'linewidth',2.5, 'markersize',10);
ylabel('height [m]', 'fontsize', 16)
legend('s_x', 's_y', 'at this time the modul hits the surface', 3); % grid
xlabel('time [s]', 'fontsize', 16)
hold on
plot(xx1,yy1,'k', xx2,yy3,'k', xx1,yy4, 'k')
hold off
print -djpeg -r300 fig_DY_03_01_03
% the end of edu_UL_2013_DY_03_01_moon_landing
```

Not being happy with the above result we have to add another condition to satisfy the third requirement, namely that the horizontal component of the landing velocity has to equal to zero as well. So,

$$0 = v_0 \sin\alpha - \frac{Ft}{m}\sin\beta .$$

This equation, together with Eqs. (e) and (f), suffices for the determination of three unknowns, i.e. t, F, β. The system of equations is, however, nonlinear. The solution is lengthier, but not difficult.

The correct value β could also be found by a trial and error approach as shown in the program `D03_moon_landing.m`. For $\beta = 3.75°$ the vertical component of the landing velocity is very small as indicated in Fig. D10.

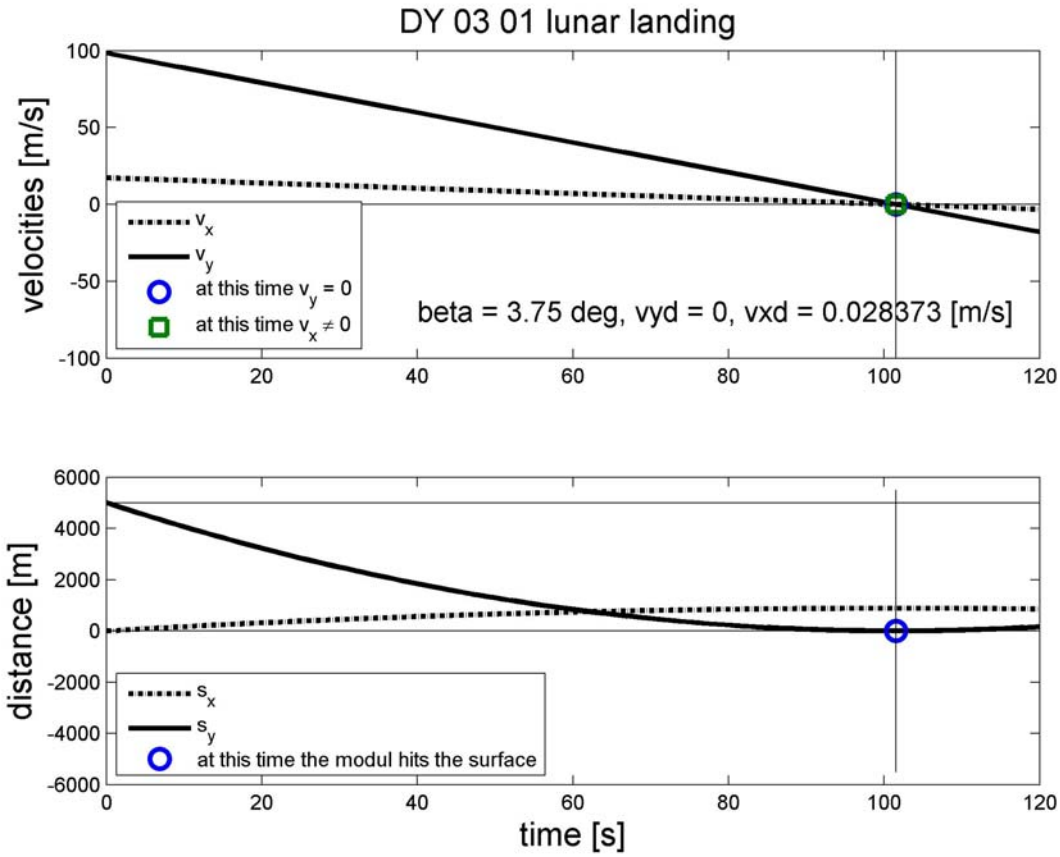


Fig. D10. Moon landing – improved

D4. Dynamics of a particle subjected to a circular motion

Let's remind what kinematics says about the motion of a particle along the circle with a radius r. If the current angular displacement is φ, then the Cartesian coordinates of a generic point can be expressed by

$$x = r\cos\varphi,\ y = r\sin\varphi, \qquad \text{(D4\_1)}$$

where the angle φ is generally a function of time. It is convenient to measure it from a suitably chosen axis counterclockwise.

Generally, the angle φ depends on the angular velocity while the angular velocity depends on time, namely $\varphi = f(\omega), \omega = g(t)$. To express the Cartesian components of velocity and acceleration of the point L as functions of time we have to evaluate the first and second derivatives of Eq. (D4_1). Thus

$$v_x = -r\omega \sin\varphi = -\omega y\,,$$
$$v_y = +r\omega \cos\varphi = +\omega x\,. \qquad \text{... (D4\_2)}$$

$$a_x = -r\omega^2 \cos\varphi - r\varepsilon \sin\varphi = -\omega^2 x - \varepsilon y\,,$$
$$a_y = -r\omega^2 \sin\varphi + r\varepsilon \cos\varphi = -\omega^2 y + \varepsilon x\,. \qquad \text{... (D4\_3)}$$

The above relations are simplified if $\omega = const$, because it that case $\varepsilon = 0$.

Often, the analysis is provided using not Cartesian but polar coordinates, that are defined in the tangent (t) and the normal (n) directions. For magnitudes of vector quantities $\vec{v}, \vec{a}$ we could write

$v = r\omega$	... velocity which has always the tangential direction,
$a_{\mathrm{t}} = r\varepsilon$	... tangential component of acceleration,
$a_{\mathrm{n}} = r\omega^2$	... normal, or centripetal, component of acceleration,
$a = r\sqrt{\omega^4 + \varepsilon^2}$	… magnitude of resulting acceleration $\vec{a} = \vec{a}_t + \vec{a}_n$.

Example – a particle moving along a circular trajectory

Given: R, m, f_1, f_2. At the beginning, i.e. for $\varphi = 0$, the particle has an initial tangent velocity v_0. Assume that there is a different coefficient of friction between the particle and cylindrical wall, say (f_1) and between the particle and the horizontal support, say (f_2). Consider the counterclockwise motion. See Fig. D11.
Determine: The location, where the particle stops.

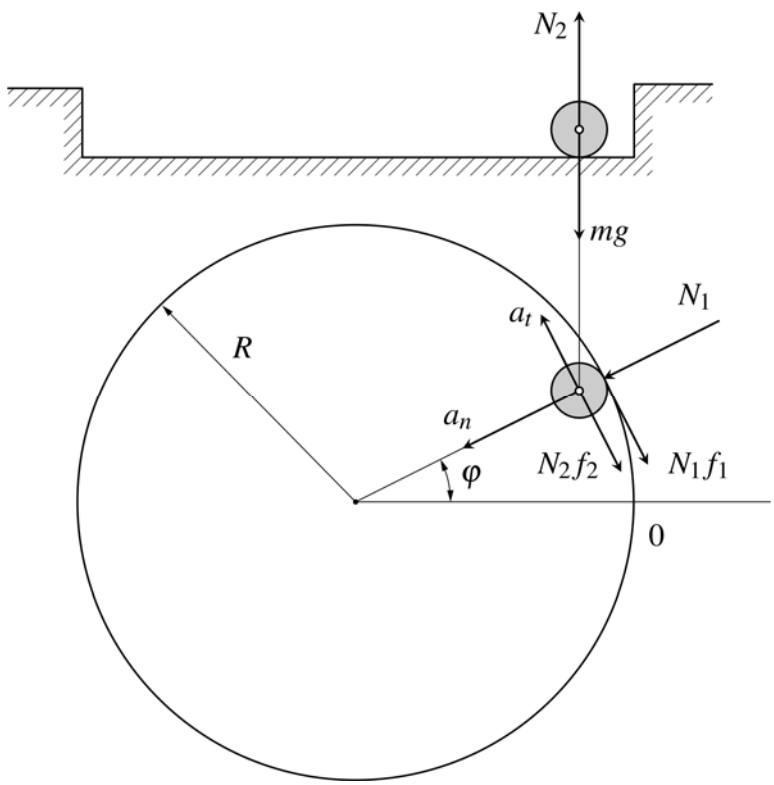


Fig. D11. A particle moving along a circular trajectory

Cylindrical coordinates are considered. Recall that the particle has a zero radius, so all the forces actually act within a single 'contact' point. The equations of motion are written in the direction of the tangent (t), and in directions of two normals, i.e. in directions of (n) and (b).

t: $ma_t = -N_1 f_1 - N_2 f_2$,

n: $ma_n = N_1$,

b: $0 = N_2 - mg$.

Kinematic relations for tangent and normal accelerations are

$$a_\mathrm{t} = R\ddot{\varphi} = R\varepsilon\,, a_\mathrm{n} = R\dot{\varphi} = R\omega^2 = \frac{v^2}{R} = R\ddot{\varphi}\,.$$

The angular acceleration can be expressed as $\varepsilon = \dfrac{\mathrm{d}\omega^2}{2\mathrm{d}\varphi}$.

Using the above kinematic relations, extracting the reaction forces and substituting them into the equation of motion, written for the tangent direction, we get

$$ma_t = -ma_\mathrm{n} f_1 - mgf_2\,,$$

$$R\varepsilon = -R\omega^2 f_1 - gf_2\,,$$

$$\frac{\mathrm{d}\omega^2}{2\mathrm{d}\varphi} = -\omega^2 f_1 - \frac{g}{R} f_2\,.$$

We are looking such a value of angular coordinate, say φ_c, where the particle stops, i.e. for the moment when the angular velocity reaches the zero value, i.e. $\omega = 0$. Integrating the last equation in proper limits we get

$$\int_{\omega_0}^{0} \frac{\mathrm{d}\omega^2}{\omega^2 f_1 + \frac{g}{R} f_2} = -2\int_0^{\varphi_\mathrm{c}} \mathrm{d}\varphi\,,$$

$$\frac{1}{f_1}\left[\ln\left(\omega^2 f_1 + \frac{g}{R} f_2\right)\right]_{\omega_0}^{0} = -2\varphi_\mathrm{c}\,,$$

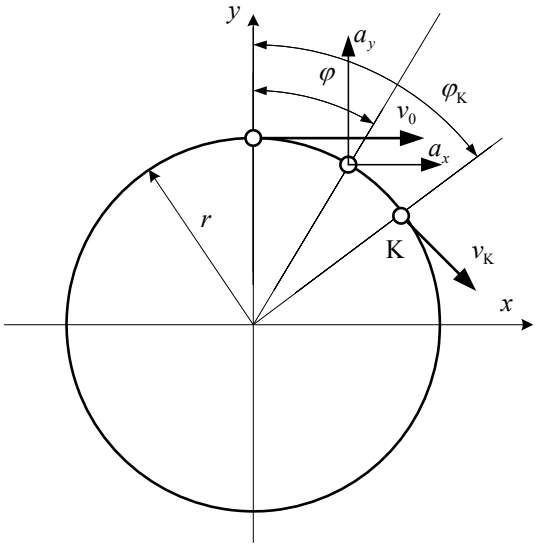
$$\frac{1}{f_1}\left[\ln\left(\omega_0^2 f_1 + \frac{g}{R} f_2\right)\right] - \frac{1}{f_1}\ln\left(\frac{g}{R} f_2\right) = 2\varphi_\mathrm{c}\,,$$

$$\varphi_\mathrm{c} = \frac{1}{2f_1}\ln\frac{\omega_0^2 f_1 + \frac{g}{R} f_2}{\frac{g}{R} f_2} = \frac{1}{2f_1}\ln\left(\frac{\omega_0^2 R f_1}{g f_2} + 1\right)\text{, where } \omega_0 = \frac{v_0}{R}\text{, so}$$

$$\varphi_\mathrm{c} = \frac{1}{2f_1}\ln\left(\frac{v_0^2 f_1}{R g f_2} + 1\right).$$

Example – from the motion along an outer surface of a cylinder to the first cosmic velocity

Given: r, m, v_0. A particle of mass m, being at the beginning on the top of a cylinder of the radius r, is released with horizontal velocity v_0. Friction is neglected. See Fig. D12.
Determine: The release and hit points.



The equations of motion for the first part, i.e. from initial position to the point K, where the particle loses its contact with the cylinder, written for a generic position denoted by angle φ, are written in the direction of tangent and normal directions, respectively

$$\text{t}: \quad ma_{\text{t}} = mg\sin\varphi\,, \qquad \text{(a)}$$

$$\text{n}: \quad ma_{\text{n}} = mg\cos\varphi - N\,. \qquad \text{(b)}$$

Fig. D12. Motion along a cylinder surface and consecutive oblique throw

Eq. (a) indicates that the tangential component of acceleration is independent of the particle mass.

Kinematic relations are

$$a_{\text{t}} = r\varepsilon\,, \qquad a_{\text{n}} = r\omega^2\,, \qquad \varepsilon = \frac{\mathrm{d}\omega^2}{2\mathrm{d}\varphi}\,. \qquad \text{(c)}$$

Rearranging Eq. (a) we get

$$r\frac{\mathrm{d}\omega^2}{2\mathrm{d}\varphi} = g\sin\varphi\,. \qquad \text{(d)}$$

We start with $\varphi = 0$ and with initial angular velocity $\omega_0 = \frac{v_0}{r}$. Let the release point is indicated by so far unknown angle φ_{k} – the corresponding angular velocity is ω_{k}. Integrating Eq. (d) we get

$$\int_{\omega_o^2}^{\omega_k^2} \mathrm{d}\omega^2 = \frac{2g}{r}\int_0^{\varphi_{\text{k}}} \sin\varphi\,\mathrm{d}\varphi\,,$$

$$\omega_{\text{k}}^2 - \omega_0^2 = -\frac{2g}{r}\left[\cos\varphi\right]_0^{\varphi_{\text{k}}}\,,$$

$$\omega_{\text{k}}^2 - \omega_0^2 = -\frac{2g}{r}\left[1 - \cos\varphi_{\text{k}}\right]\,,$$

$$\omega_{\text{k}}^2 = \omega_0^2 + \frac{2g}{r}\left[1 - \cos\varphi_{\text{k}}\right]\,. \qquad \text{(c)}$$

So far we know neither ω_{k} nor φ_{k}. The relation between them is obtained from Eq. (b). Also, we know that in the moment of release the normal reaction N should attain the zero value. So,

$m(a_{\mathrm{n}})_{\mathrm{k}} = mg\cos\varphi_{\mathrm{k}}$, where $(a_{\mathrm{n}})_{\mathrm{k}} = r\omega_k^2$

and consequently

$$\omega_{\mathrm{k}}^2 = \frac{g}{r}\cos\varphi_{\mathrm{k}}. \qquad \text{(d)}$$

Substituting (d) into (c) we get

$$\frac{g}{r}\cos\varphi_{\mathrm{k}} = \omega_0^2 + \frac{2g}{r}\left[1-\cos\varphi_{\mathrm{k}}\right],$$

The cosine of the release angle is

$$\cos\varphi_{\mathrm{k}} = \frac{r\omega_0^2}{3g} + \frac{2}{3}.$$

We know that $\omega_0 = v_0 / r$, so

$$\cos\varphi_{\mathrm{k}} = \frac{v_0^2}{3rg} + \frac{2}{3}. \qquad \text{(e)}$$

Solvability condition. The angle φ_{k} have to be a real number, so the argument of arcus cosine function have to be less or equal to one. On the edge of solvability we have

$$\frac{v_0^2}{3rg} + \frac{2}{3} = 1. \qquad \text{(f)}$$

Considering for a moment the quantity r to be the Earth radius, i.e. $r = 6378000\,\mathrm{m}$, and taking the gravitational acceleration as $g = 9.81\,\mathrm{m/s^2}$, we get

$$v_0 = \sqrt{rg} = 7{,}91\,\mathrm{km/s}.$$

On the verge of solvability, we surprisingly obtained the first cosmic velocity – the initial horizontal velocity needed for a particle to circle the Earth and never fell to the ground.

And now, back to our task.

For the zero initial velocity, there will be no motion. For infinitesimally small velocity the particle starts to move ‘downward’. The release point then will be computed from Eq. (e). We get

$\cos\varphi_{\mathrm{k}} = \dfrac{2}{3} \quad \Rightarrow \varphi_{\mathrm{k}} = \arcsin\dfrac{2}{3} = 0.7297$. This, expressed in degrees, is 45.0280.

The corresponding tangent velocity at the point K would be $v_{\mathrm{K}} = r\omega_{\mathrm{K}}$. Using the relation (d) we obtain

$$v_{\mathrm{K}} = \sqrt{rg\cos\varphi_{\mathrm{K}}}\ .$$

From now on, we can solve the standard ballistic problem for a particle being shot from point K with the initial velocity v_{K} and look for the impact or contact point D.

The magnitude of initial velocity is $v_{\mathrm{K}} = r\omega_{\mathrm{K}}$ – its direction is in a tangent line at the point K to the surface. The initial location has the coordinates

$$x_0 = r\sin\varphi_{\mathrm{K}}\ ,$$
$$y_0 = r\cos\varphi_{\mathrm{K}}\ .$$

The components of initial velocity are

$$v_{0x} = v_{\mathrm{K}}\cos\varphi_{\mathrm{K}}\ ,$$
$$v_{0y} = v_{\mathrm{K}}\sin\varphi_{\mathrm{K}}\ .$$

Equations of motion and their solution

$$ma_x = 0\ , \qquad ma_y = -mg\ ,$$

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = 0\ , \qquad \frac{\mathrm{d}v_y}{\mathrm{d}t} = -g\ ,$$

$$\int_{v_{0x}}^{v_x} \mathrm{d}v_x = 0\ , \qquad \int_{-v_{0y}}^{v_y} \mathrm{d}v_y = -g\int_0^t \mathrm{d}t\ ,$$

$$v_x = v_{0x} = v_{\mathrm{K}}\cos\varphi_{\mathrm{K}} = const\ , \qquad v_y = -v_{0y} - gt = -v_{\mathrm{K}}\sin\varphi_{\mathrm{K}} - gt\ ,$$

$$\frac{\mathrm{d}s_x}{\mathrm{d}t} = v_x\ , \qquad \frac{\mathrm{d}s_y}{\mathrm{d}t} = v_y\ ,$$

$$\int_{x_0}^{s_x} \mathrm{d}s_x = \int_0^t v_x\mathrm{d}t\ , \qquad \int_{y_0}^{s_y} \mathrm{d}s_y = \int_0^t v_y\mathrm{d}t = \int_0^t \left(-v_{0\mathrm{y}} - gt\right)\mathrm{d}t\ ,$$

$$s_x - x_0 = v_x t = v_{0x}t\ , \qquad s_y - y_0 = -v_{0y}t - \frac{1}{2}gt^2\ .$$

Parametric equations of the particle trajectory are

$$s_x = x_0 + v_{0x}t\ , \qquad s_y = y_0 - v_{0y}t - \frac{1}{2}gt^2\ .$$

The impact point is determined from the condition that $s_y = 0$. So

$$y_0 - v_{0y}t - \frac{1}{2}gt^2 = 0$$

and finally

$$t_{1,2} = \frac{b^2 \pm \sqrt{b^2 - 4ac}}{2a}, \text{ where } \quad a = \frac{g}{2}, b = v_{0y}, c = -y_0 .$$

Only the positive root is meaningful.

For details see the program `D04_circular_orbit`. The results are graphically depicted in Fig. D13.

```
% D04_circular_orbit
 mtl_DY_02_04
clear
r = 2; v0 = 5; g = 9.81;    % input data
omega = v0/r;               % initial angular velocity
arg = omega^2/(3*g) + 2/3;
fik = acos(arg); fik_deg = fik*180/pi;      % release angle
xk = r*sin(fik); yk = r*cos(fik);           % point of release
omegak = omega^2 + 2*g*(1 - cos(fik))/r;    % release angular velocity
vk = r*omegak;                              % release tangential velocity
t_range = 0:0.05:0.25;                      % time range

% initial conditions for the second part of motion
x0 = r*sin(fik);        y0 = r*cos(fik);
v0x = vk*cos(fik);      v0y = vk*sin(fik);

% the second part of the motion
% trajectory of the free fall with prescribed initial conditions
it = 0;
for t = t_range
    it = it + 1;
    sx(it) = x0 + v0x*t;                        % (a)
    sy(it) = y0 - v0y*t - 0.5*g*t^2;            % (b)
end

% hit point is defined by sy = 0;
% express t from (b), which leads to quadratic equation
% only plus root is applicable in this case
a = 0.5*g; b = v0y; c = -y0;
td = (-b + sqrt(b^2 - 4*a*c))/(2*a);
% alternatively, use matlab roots function
rr = roots([a,b,c]);        % the second root has no meaning
% and substitute it into (a)
sxd = x0 + v0x*td;
sxd2 = x0 + v0x*rr(1);      % imaginary hit point

% plot it
txt1 = ['v0 = ' num2str(v0) ' m/s  \phi_K = ' num2str(fik_deg) ' deg'];
txt2 = ['sxd = ' num2str(sxd) ' m'];    txt3 = ('\phi_K');
% auxiliary lines
xx1 = [-r 4.2];  yy1 = [0 0];
xx2 = [0 0];  yy2 = [0 r];
xx3 = [0 xk]; yy3 = [0 yk];
figure(1)
fi = 0:pi/90:2*pi;
x = r*sin(fi); y = r*cos(fi);
plot(x,y,'k-', xk,yk,'ko', sx,sy,'k--', xx1,yy1,'k-.', xx2,yy2,'k:', ...
    xx3,yy3,'k:', sxd,0, 'sk', ...
    'linewidth', 2, 'markersize', 10)
legend('circle', 'release point', 'free fall trajectory', 'ground', ...
   'line1', 'line2', 'hit point', 3)
axis('equal'); title('DY 02 04')
text(1.7,-1.8,txt1); text(1.7,-2,txt2); text(0.15,1,txt3)
print -djpeg -r300 fig_DY_02_04_02
```

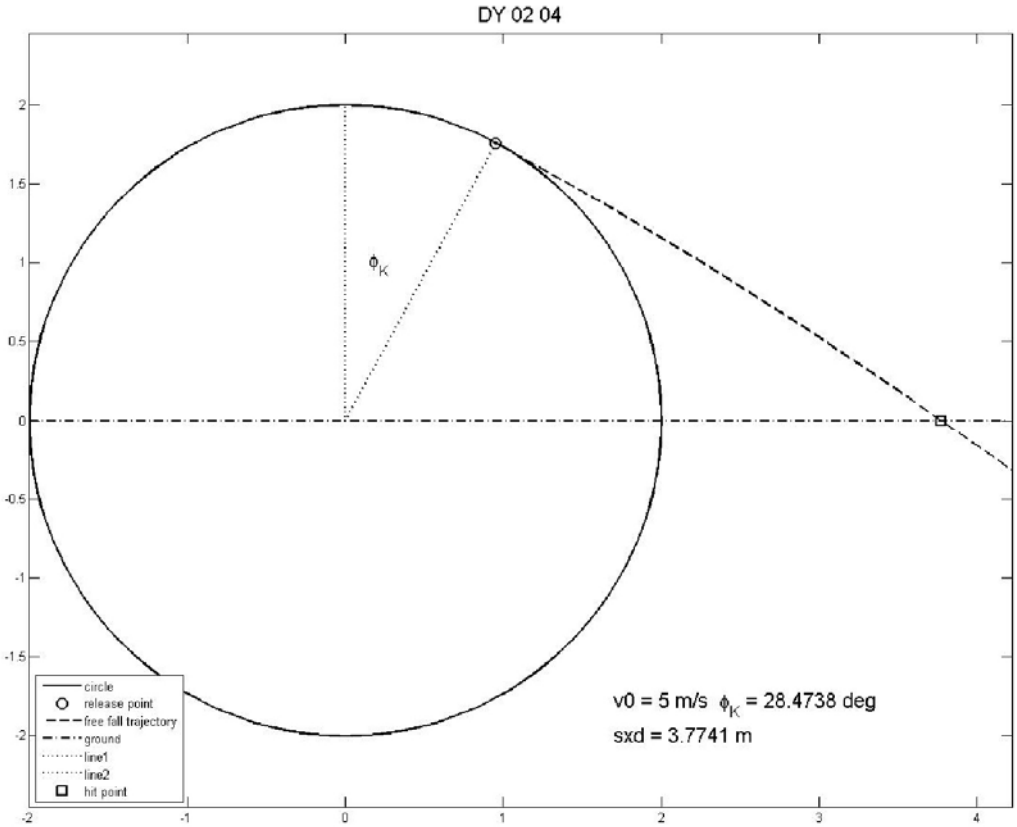


Fig. D13. Matlab output

D5. Newton's and d'Alembert's formulations of equations of motion

The fact that the product $m\mathbf{a}$, describing the inertia force, has the dimension of force, was used by d'Alembert for introducing so-called *apparent inertia force* in the form

$$\mathbf{D} = -m\mathbf{a}. \qquad \text{(D5\_1)}$$

This allows rewriting the Newton equation of motion $m\mathbf{a} = \sum \mathbf{F}_i$ into the form

$$\mathbf{D} + \sum \mathbf{F}_i = 0. \qquad \text{(D5\_2)}$$

So-called d'Alembert's principle states that the apparent inertia forces and other acting forces are in the state of 'dynamic equilibrium'.

This is, however, an apparent equilibrium characterized by the fact that time in our minds is temporally frozen. For a given moment we might consider the solved task as an apparent 'equilibrium' case. In the following moment we also have an apparent equilibrium, but a different one. In an inertial frame of reference, the d'Alembert formulation is just a simple mathematical reformulation of the classical Newton's formulation of the equation of motion requiring to take inertia force and shift it to the other side of the equation with an opposite sign and call it the *apparent inertia force*. In a non-inertial frame of reference, it is not so simple. We will show that it is the observer's point of view that plays a crucial role.

Inertial and non-inertial frames (systems) of reference

The inertial system is a useful engineering approximation defined by the assumption that it is a system which is stationary with respect to fixed stars[5]. The non-inertial system moves with acceleration with respect to an inertia system. In many engineering application, an observer might safely consider the Mother Earth as an inertial system. A coordinate system firmly connected to a rotating merry-go-round is a good example of a non-stationary system of reference.

An example, which might shed light on the difference

Let us examine a simple example. Consider a stone being whirled around on a string, in a horizontal plane. The effect of gravity is neglected here. There are two alternative and equivalent ways how to tackle the problem – using either Newton's or d'Alembert's formulations. See Fig. D14.

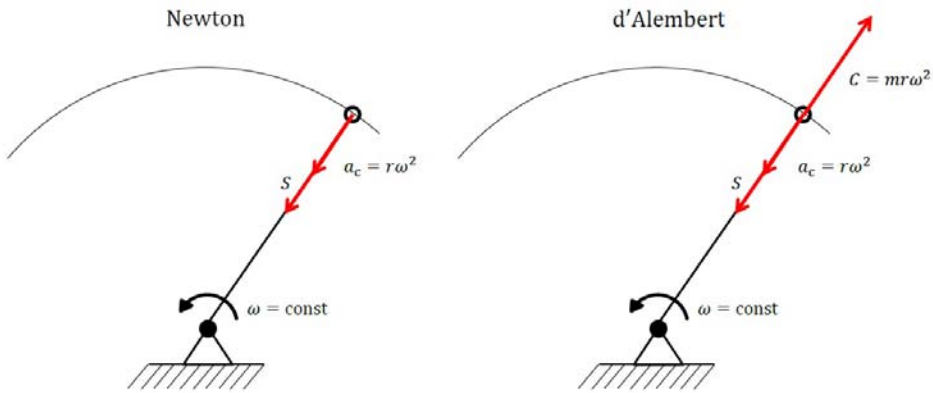


Fig. D14. Observer's view – Newton and d'Alembert

Newton's formulation, i.e. the equation of motion in the form $ma_c = S$, with a_c being the normal or centripetal acceleration, is applicable for an observer in an inertial frame of reference, for somebody who is located at a fixed point of the Universe. Newton's second law has a form of equivalence of forces. For a mass particle, it states that a product of mass and acceleration is equal to the sum of acting forces. Newton calls the force on the left-hand side of the equation, i.e., ma_c, the *inertial force*, while the constraint force, the force in the string, i.e. S, he denotes by the term centripetal force. We have to pull on the string to keep the stone in the circle. In this case, both forces have the same direction and the same magnitude, but they are not identical. They are of different origins.

D'Alembert's formulation is applicable for an observer in the non-inertial frame of reference, for somebody, who is sitting on the rotating particle. D'Alembert showed that one can write equations of motion by means of equivalent, seemingly static, equilibrium equations, by adding the so-called *apparent*[6] *inertial force*. Generally, the apparent inertial force is a product of mass and negative acceleration. See [1], [2].

[5] Of course, we know that the Universe is expanding and constantly accelerating. So, there are no fixed stars available and generally, no inertial frame of reference exists. Nevertheless, the Earth can be for many engineering applications approximately considered as the inertial frame of reference since its orbital accelerations, due to Earth's daily and annual rotations are small.

[6] Other terms used for the adjective 'apparent' are d'Alembert, fictitious and pseudo-force. See [1].

To use Newton's law correctly in a non-inertial frame, the apparent inertial force must be added. In this case, the apparent force is the **centrifugal force** $\vec{C} = -m\vec{a}_c$, where $a_c = |\vec{a}_c| = r\omega^2$ is the magnitude of the **centripetal acceleration.** See [2], [3]. D´Alembert's formulation has a form of the equilibrium of forces – meaning that the sum of all forces is equal to zero. In scalar notation, where arrows indicate the direction of accelerations and forces, we might write, $S - C = 0$, where $C = mr\omega^2$ is the magnitude of the centrifugal force. See Fig. D14. Notice that the centrifugal force $\vec{C}$, as a vector, has an opposite sign with respect to the vector of centripetal acceleration $\vec{a}_c$. The corresponding scalar equation $S - C = 0$ comes from the idea of the free-body-diagram reasoning, which is based on the idea of replacing the effects of constraints by equivalent forces – in this case, the string is mentally cut and replaced by an equivalent force, say S. To an observer sitting on the rotating particle, the centrifugal force appears to be the external force – not 'apparent' at all[7].

Newton's and d´Alembert's formulations, written in scalar notations with directions of forces indicated by arrows in Fig. D14, are

$$\text{Newton}: \; ma_c = S \quad \text{and} \quad \text{d'Alembert}: \; S - C = 0, \quad \text{where} \quad C = ma_c, \, a_c = r\omega^2 .$$

Equations describing the motion, lead to same conclusions, are seemingly identical but have a completely different background, so

$$mr\omega^2 = S \quad \Leftrightarrow \quad S - mr\omega^2 = 0 .$$

To summarize briefly:
In Newton's formulation, the term ma_c is the inertial force. In d'Alembert's formulation, the variable C is the centrifugal force. In both formulations, the term a_c is the normal or centripetal acceleration and the constraint force, denoted S, is the force in the string.

To summarize at length:
For an observer in the inertial frame of reference, who is using Newton's formulation, the product of mass and acceleration should be called the *inertial force*. Calling it the centripetal force is misleading because this term is usually reserved for the constraint force. It should be emphasized that for an observer in the inertial frame of reference the term centrifugal force has no meaning.

An observer in the non-inertial frame of reference, who is using the d'Alembert's formulation, and writes dynamical equations of equilibrium, has to add **apparent inertial forces** to existing external forces. Apparent inertial forces are defined as a product of mass and negative acceleration of the non-inertial frame. These apparent inertial forces seemingly arise out of nothing – yet they do have a sound origin based on the transformation of coordinates between the stationary (inertial) and accelerating (non-inertial) frames of reference. In our example with the rotating particle, the role of the apparent inertial force is played by the centrifugal

[7] The problem is a little bit obscured by two contradictory meanings of the adjective 'apparent'. In the Webster dictionary, you might find two sentences with opposite explanations. In the first sentence 'He is apparently rich' it is understood that his richness is obvious, clearly visible, nobody doubts it. In another example, the term 'apparent horizon' is used as an antonym to the 'real horizon'.

force, while the external force is the constraint force in the string. The term *apparent* is used by stationary observers. For observers in the accelerating frame of reference, the centrifugal force can be felt and could be measured and appears not apparent but quite real. This might be a source of confusion.

One of the reasons leading to this confusion of tongues is due to the ambiguous terminology used for the description of the rotation of bodies in mechanics. Not only are the same terms often used for different kind of forces, sometimes different terms describe identical forces. In addition, confusion might also arise because of two possible observation points. These are either from the stationary inertial frame of reference or from the accelerating – i.e. the non-inertial – frame of reference. Newton's and d'Alembert's formulations are proper tools corresponding to these two viewpoints and lead to identical results.

Generally, there are other apparent forces, such as Euler, centrifugal and Coriolis forces which are proportional to negative tangential, centripetal and Coriolis accelerations respectively that were thoroughly treated and explained in kinematics.

To be clear and consistent, we should distinguish the terms *inertial force* and *apparent inertial force*. The ***inertial force* is a product of mass and acceleration**. The ***apparent inertial force* is a product of mass and negative acceleration**. Not many authors observe this simple terminological rule and in the latter case, the adjective *apparent* is often dropped. Regrettably, the term inertial force is used for whatever meaning a particular author finds suitable. Compare [2] and [6].

In Newton's Principia the term **centripetal force is reserved for the external forces, which might be of different origins** – a constraint force, gravitational force, magnetic force, etc. In this respect, most publications follow that lead, but at the same time they often claim that the centrifugal force is ma_c. According to Newton's terminology, the product of mass and acceleration ma_c is the inertial force, not the centrifugal force. **The inertial force and the centripetal (constraint) force have the same size and direction but are distinct in nature and not identical.**

Example – mathematical pendulum

For a mathematical pendulum, see Fig. D15, consisting of a particle of mass m, swinging on the rope of length l, the equations of motion, written into tangent and normal directions, are

$$D_t + mg\sin\varphi = 0,$$

$$D_n + mg\cos\varphi - S = 0.$$

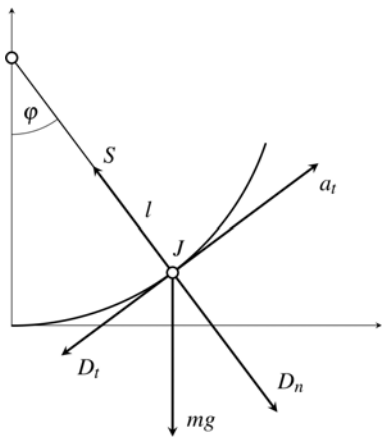


Fig. D15. Mathematical pendulum – FBD

The rope force is denoted S, while the apparent tangential inertia force, whose direction is opposite to the assumed positive tangential acceleration, is

$$D_t = ma_t = ml\varepsilon = ml\ddot{\varphi}.$$

The apparent normal force, also called the centrifugal force, is denoted *O*. Its direction is opposite the assumed positive normal (centripetal) acceleration

$$D_{\mathrm{n}} = O = ma_{\mathrm{n}} = ml\omega^2 = ml\dot{\varphi}^2 .$$

D'Alembert's formulation states that the vector sum of all the forces is equal to zero. In the scalar notation, we have

$$ml\ddot{\varphi} + mg\sin\varphi = 0 ,$$
$$ml\dot{\varphi}^2 + mg\cos\varphi - S = 0 .$$

These non-linear equations are frequently linearized for small displacement angles, i.e. for $\varphi < 5°$, assuming that

$$\sin\varphi \to \varphi, \cos\varphi \to 1 .$$

Then, we have two linear ordinary differential equations of the second order with constant coefficients, instead. The first equation is then rewritten into

$$l\ddot{\varphi} + g\varphi = 0 ,$$
$$\ddot{\varphi} + \frac{g}{l}\varphi = 0 ,$$
$$\ddot{\varphi} + \omega^2\varphi = 0, \quad \text{where} \quad \omega = \sqrt{\frac{\mathrm{g}}{\mathrm{l}}} \quad \text{is so-called angular frequency.}$$

It should be reminded that $\omega = 2\pi f = \frac{2\pi}{T}$, where $f\,[1/\mathrm{s}]$ is the circular frequency having dimension $[1/\mathrm{s}]$ which is also denoted $[\mathrm{Hz}]$. The quantity $T\,[\mathrm{s}]$ is called period.

What is the period of a one-meter pendulum? It comes from the relation $T = 2\pi\sqrt{\frac{l}{g}}$.

From Matlab we get

```
>> l = 1; g = 9.81; T = 2*pi*sqrt(l/g); T = 2.0061.
```

The result is in seconds.

Example – pendulum in accelerating lift, a steady state solution.

Imagine a pendulum in the lift which is ascending with a constant acceleration a_v. In this case the positive direction of acceleration a_v goes against the positive direction of the gravitational acceleration g. See Fig. D16.

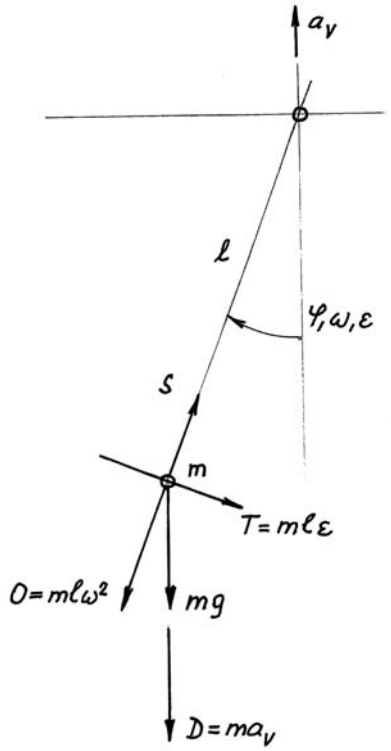


Fig. D16. Pendulum in an ascending lift

Two scalar component equations of motion, written in tangent and normal accelerations, are

$$t: \quad T + m(g + a_v)\sin\varphi = 0,$$
$$n: \quad O - S + m(g + a_v)\cos\varphi = 0.$$

Kinematics relations are

$$\varepsilon = \ddot{\varphi}, \quad \omega = \dot{\varphi}.$$

The apparent inertia forces, i.e. the tangential and normal (centrifugal) forces, are

$$T = ml\varepsilon, \qquad O = ml\omega^2.$$

The equation of motion could be alternatively obtained by writing a moment equation of motion about the centre of rotation. Thus,

$$Tl + m(g + a_v)l\sin\varphi = 0.$$

Rearranging we get

$$ml^2\varepsilon + m(g + a_v)l\sin\varphi = 0,$$
$$ml^2\ddot{\varphi} + m(g + a_v)l\sin\varphi = 0,$$
$$\ddot{\varphi} + \frac{g + a_v}{l}\sin\varphi = 0,$$
$$\ddot{\varphi} + \Omega^2\sin\varphi = 0.$$

For small angular displacements, we approximately assume $\sin\varphi \to \varphi$ and thus

$$\ddot{\varphi} + \Omega^2\varphi = 0.$$

The period of vibration is $T = \dfrac{2\pi}{\Omega} = 2\pi\sqrt{\dfrac{l}{g + a_v}}$.

Discussion for pendulum clocks using a pendulum, as the timekeeping element.

- Increasing the value of the acceleration a_v, the period of vibration decreases. The corresponding pendulum clock in the lift gains time.
- If the lift is descending with acceleration $a_v < 0$, the period of vibration increases and the pendulum clock in the lift loses time.
- In the falling lift, i.e. for $a_v = -g$, the period of vibrations increases above all limits – the pendulum actually, stops.

Conclusion: A pendulum clock in the falling lift is not a suitable time keeping device.

Example – ascending lift, a transient solution

Again, consider an ascending lift with a constant acceleration a_v. But in this case, we intend to analyze not a steady state as before, described by $a_v = const$, but a transient process. That is what happens when the lift starts to accelerate from zero initial conditions until it reaches steady state conditions. See Fig. D17.

Imagine a person, whose mass is $m = 80\ \text{kg}$, standing on a bathroom spring scale. We usually, and rather imprecisely, say that the mass of a body is measured by weighing. On spring scales we actually measure not the mass but the weight of the body, balancing it against a force in the spring which is damped.

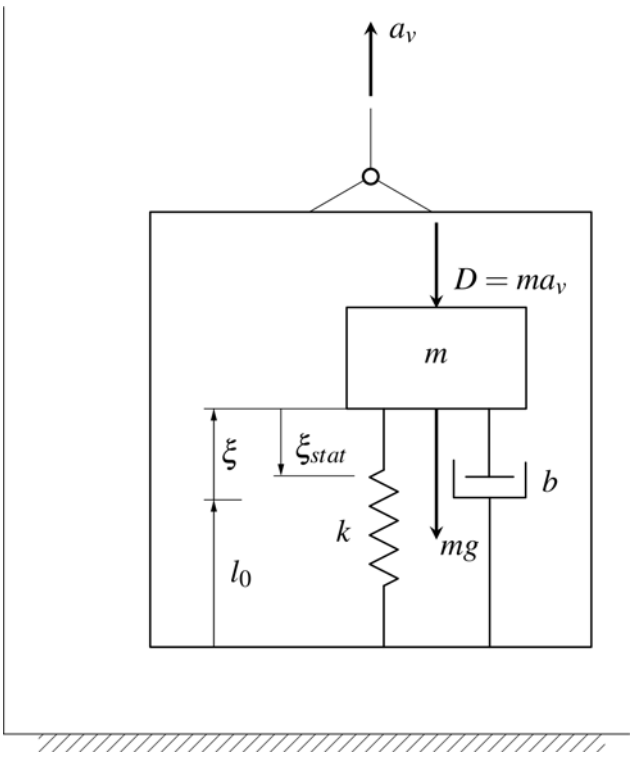


Fig. D17. Vibrating scale in an ascending lift

What is actually measured is the spring deflection and knowing the spring stiffness and the local value of gravitational acceleration, we can associate the deflection with the force and then to assign the force in the spring to the mass value by a process of linear calibration. So, using a spring balance we are actually measuring the gravitational force of a body in *newtons* but are observing the dials calibrated in mass units in *kilograms* instead. Everything works well if the weighing process is carried out in a stationary frame of reference. But the accelerating lift is a typical example of a non-stationary frame of reference.

When the lift is stationary, the person's rest weight is mg. You should not be confused by the fact that the balance is calibrated not in [N] but in [kg]. After the lift starts to accelerate upward the person's actual weight (this is what the spring balance actually measures) for a short time temporally increases and then it subsequently returns to a certain stationary value. Of course, its mass of our person does not change at all.

In this example, the stiffness of our spring balances k is chosen in such a way that $k\xi_{\text{stat}} = mg$, where ξ_{stat} is a static deflection of the spring due to the loading by the person's weight mg in a stationary lift.

The spring balance is idealized by a vibrating system with one degree of freedom consisting of the particle of mass m, the spring with stiffness k and of the linear damper, characterized by a parameter b. See Fig. D17. The equation of motion for our simplified system in a stationary frame of reference, i.e. in the lift which stays in rest or moves with a constant velocity, is described by the ordinary differential equation of the second order with constant coefficients in the form

$$m\ddot{\xi} + b\dot{\xi} + k\xi = mg\,. \qquad \text{(a)}$$

For a lift ascending upwards with acceleration a_{v} we have to add an apparent inertia force whose magnitude is $D = ma_v$ and is directed downwards. See arrows in Fig. D17, so

$$m\ddot{\xi} + b\dot{\xi} + k\xi = mg + ma_v\,. \qquad \text{(b)}$$

Rearranging we get

$$\ddot{\xi} = -\frac{b}{m}\dot{\xi} - \frac{k}{m}\xi + g + a_v\,. \qquad \text{(c)}$$

To find out what would be the weight of a person standing on the spring balance in the accelerating lift we have to evaluate the deflection ξ, relate it to $k\xi$, which is actually related to the current weight. These quantities are evidently functions of time. To solve for them the differential equation (b) has to be integrated.

For this purpose, Matlab integration procedures will be utilized. They require that the differential equation to be solved have the form of the first order differential equations.

We start by introducing

$$\ddot{\xi} = \dot{z} \Rightarrow \dot{\xi} = z\,.$$

Then, the Eq. (c) becomes

$$\dot{z} = -\frac{b}{m}z - \frac{k}{m}\xi + g + a_{\text{v}}\,.$$

Also, we introduce

$$\dot{y}_1 = \dot{\xi} \Rightarrow y_1 = \xi\,,$$
$$\dot{y}_2 = \dot{z} \Rightarrow y_2 = z\,.$$

So, instead of one differential equation of the second order, we have two differential equations of the first order in the form

$$\dot{y}_1 = y_2 \ ,$$
$$\dot{y}_2 = -\frac{b}{m} y_2 - \frac{k}{m} y_1 + g + a_\mathrm{v} \, .$$

See the program `D05_accelerating_lift_transient_solution`, and its output presented in Fig. D18.

.

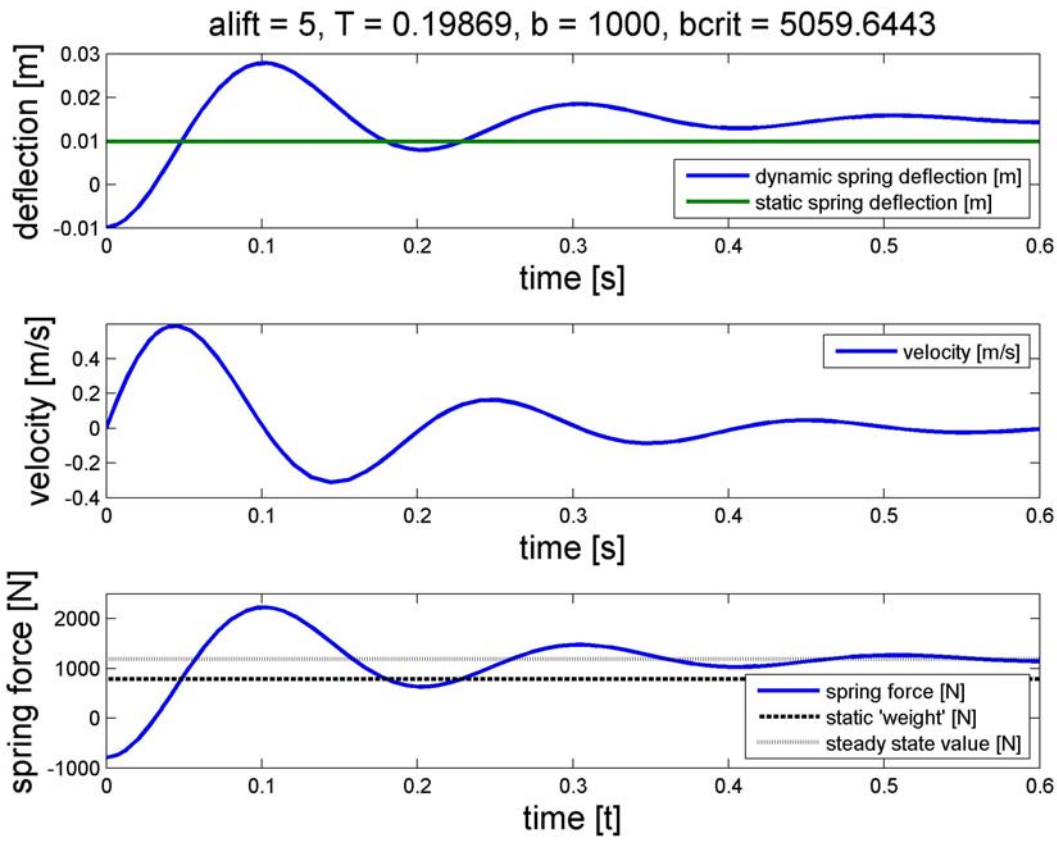


Fig. D18. Matlab output – vibrating bathroom scale in an accelerating lift

```
D05_accelerating_lift_transient_solution
% original file name is edu_UL_2013_DY_03_vaha_ve_vytahu_en
%
% program requires procedure function dy = vaha(t,y)
%
clear
global av k m g B;
av = 5;                      % upward acceleration
k = 80000;                   % spring stiffness
m = 80;                      % mass of person standing on balances
g = 9.81;                    % gravitational acceleration
b = 1000;                    % damping coefficient
omega = sqrt(k/m);           % frequency
B = b/m;
T =2*pi/omega;               % period
ksi_stat = m*g/k;            % static deflection
ksi_dyn_ust = m*(av + g);    % steady-state level
t_span = [0 0.6];            % time span
```

```
y0 = [-ksi_stat 0];          % initial conditions

b_krit = 2*m*sqrt(k/m)       % critical damping

[t,y] = ode23('vaha', t_span, y0);

ksi = y(:,1);
S = k*ksi;

xx = [0 0.6];
yy = [ksi_stat ksi_stat];
yyy = [m*g m*g];
yyyy = [ksi_dyn_ust ksi_dyn_ust];

figure(1)
subplot(3,1,1)
plot(t,y(:,1), xx,yy,'linewidth', 2)
legend('dynamic spring deflection [m]', 'static spring deflection [m]', 4)
lab = ['alift = ' num2str(av), ', T = ' num2str(T) ', b = ', num2str(b), ', bcrit = '
num2str(b_krit)];
title(lab, 'fontsize', 16); axis([0 0.6 -0.01 0.03]);
xlabel('time [s]', 'fontsize', 16); ylabel('deflection [m]', 'fontsize', 16)
subplot(3,1,2)
plot(t,y(:,2),'linewidth', 2)
legend('velocity [m/s]'); axis([0 0.6 -0.4 0.6])
xlabel('time [s]', 'fontsize', 16); ylabel('velocity [m/s]', 'fontsize', 16)
subplot(3,1,3)
plot(t,S, xx,yyy,'k--', xx,yyyy,'k:','linewidth', 2)
ylabel('spring force [N]', 'fontsize', 16)
xlabel('time [t]', 'fontsize', 16); axis([0 0.6 -1000 2500])
legend('spring force [N]', 'static ''weight'' [N]', 'steady state value [N]', 4)
print -djpeg -r300 fig_vaha_ve_vytahu_en
% end of edu_UL_2013_DY_03_vaha_ve_vytahu_en

function dy = vaha(t,y)
global av k m g B;
dy = zeros(2,1);
dy(1) = y(2);
dy(2) = -B*y(2)-k*y(1)/m + av + g;
% end
```

One sees, that the transient solution converges to the stationary one. This is reminded by the following example.

Example – ascending lift, a steady state solution

Given: A person, considered as a particle of mass m, stands on the spring balances firmly connected to a lift, which ascends by a constant acceleration a_v.

Determine: The normal reaction N from the balances to the person. See Fig. D19.

Equation of motion is

$$-D + N - mg = 0,$$

$$D = ma_v,\ a_v = \text{const},$$

$$\mathrm{N} = \mathrm{m}(a_v + g).$$

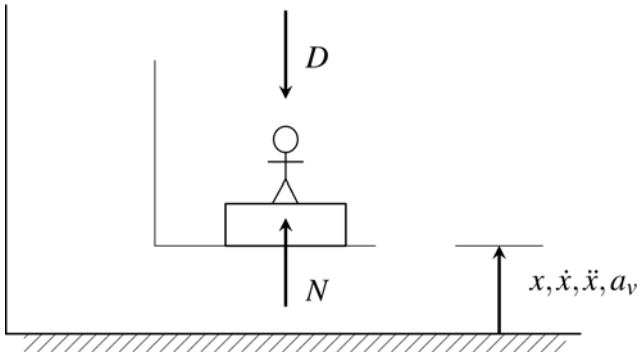


Fig. D19. Weighing a person in a lift

The normal force N represents the value of the 'apparent weight' in an ascending lift. Of course, the mass is not changed at all. If the lift falls down (i.e. $a_v = -g$), then the force

acting between the person and the balances is equal to zero and the state of weightlessness, or the absence of weight, is felt.

Extended Example A1 – the same phenomenon as viewed by inertial and non-inertial observers

Newton's law, in its simple form, i.e. $\vec{F} = m\vec{a}$, is only applicable to a particle in a so-called *inertial frame of reference*. In older textbooks, the term inertial frame reference was nicely defined as a system which is attached to fixed stars. Such a system can be either absolutely still or moving with a constant velocity with respect to fixed stars. *A non-inertial frame of reference* is a frame which undergoes acceleration with respect to an inertial frame.

Since the Universe is expanding and constantly accelerating, there are no fixed stars available and generally, no inertial frame of reference exists. Nevertheless, the Earth can be – for many (but not all[8]) engineering applications – approximately considered as the inertial frame of reference since its orbital accelerations, due to Earth's daily and annual rotations are small.

Consider a simple task where the Earth is considered as the inertial frame of reference while the streetcar, accelerating on tracks laid on the flat Earth's surface, serves as an example of a non-inertial frame.

Let the coordinate system x, y represent our approximate inertial reference system, firmly connected to the Earth. Our task is to analyze the trajectory of a particle having the mass m, being propelled by a constant force F (imagine a small rocket engine attached to the particle), which resides in the street car moving in x-direction with constant acceleration a along a straight horizontal track, while the non-inertial system of reference, i.e. x', y', is firmly connected to the accelerating streetcar. See Fig. D20. The initial velocities of the particle with respect to the street car are known. Only the planar motion is considered and also the Earth's gravity is taken into account.

Using Newton's formulation, the equations of motion, relative to the Earth, are

$$m\ddot{x} = F\cos\alpha, \quad m\ddot{y} = F\sin\alpha - mg. \quad \text{(A1\_1)}$$

$$\ddot{x} = \frac{F}{m}\cos\alpha, \quad \ddot{y} = \frac{F}{m}\sin\alpha - g.$$

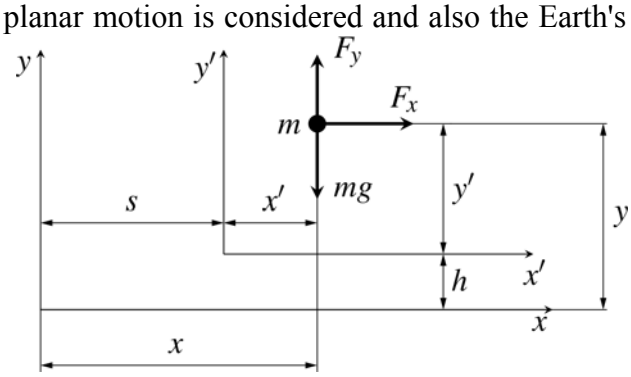


Fig. D20. A particle in an accelerating streetcar

Initial conditions of the streetcar.
At the time $t = 0$ the axes y and y' coincide, while there is a constant distance h between the axes x' and x. The street car's initial velocity is zero.

[8] Foucault's pendulum or the South-North oriented rivers or the trade winds are examples, where the Earth's cannot be considered as an inertial frame of reference, since its rotation and consequent acceleration cannot be neglected.

Initial conditions of the particle.
At the beginning, the particle resides at the origin of x', y' system and its initial velocity components are $v_{x0} = v_{x'0}, v_{y0} = v_{y'0}$.

Taking into account the prescribed initial conditions, the double integration of Eqs. (A1_1), with respect to time, gives the particle velocities and the particle coordinates as functions of time, as seen by an outside observer.

$$v_x = v_{x0} + \frac{Ft}{m}\cos\alpha, \quad v_y = v_{y0} + \frac{Ft}{m}\sin\alpha - gt. \tag{A1_2a}$$

$$x = v_{x0}t + \frac{Ft^2}{2m}\cos\alpha, \quad y = v_{y0}t + \frac{Ft^2}{2m}\sin\alpha - \frac{1}{2}gt^2. \tag{A1_2b}$$

Due to the prescribed constant acceleration of the streetcar, the transformation of the coordinates, between the Earth coordinate system and the streetcar's coordinate system, is as follows.

$$x = x' + s, \quad s = \frac{1}{2}at^2, \quad y = y' + h, \quad h = const. \tag{A1_3}$$

The velocities $v_{x'0}, v_{y'0}$ belong to the particle. Hence

$$\ddot{x} = \ddot{x}' + \ddot{s}, \quad \ddot{s} = a.$$
$$\ddot{x} = \ddot{x}' + a, \quad \ddot{y} = \ddot{y}'. \tag{A1_4}$$

Substituting the last relation of Eqs. (A1_4) into Eqs. (A1_1) gives the equations of motion of the particle relative to the accelerating street car

$$m\ddot{x}' = F\cos\alpha - ma, \quad m\ddot{y}' = F\sin\alpha - mg. \tag{A1_5}$$

The equations (A1_5) have the form of the *equivalence of forces*. The left-hand side force (the inertial force) is equal to the sum of right-hand side, i.e. external, forces. Using the d'Alembert's principle we might write the equations of motion in an alternative form

$$-m\ddot{x}' - ma + F\cos\alpha = 0, \quad -m\ddot{y}' - mg + F\sin\alpha = 0. \tag{A1_6}$$

Now, the equations of motion (A1_6) are expressed in the form of an *equilibrium of forces*. The sum of all forces is equal to zero. Of course, it is not the proper 'static' equilibrium; it is a sort of virtual equilibrium, expressed for a moment frozen in time.

An additional force i.e. $-ma$ appears on the right-hand side of equations of motion, in Eq. (A1_6), written for the non-inertial frame of reference. Cornelius Lancozs [2], calls it an *apparent force* or *d'Alembert force*, which – for an observer attached to the Earth – seemingly emerges out of nothing[9]. Evidently, it is the acceleration of the moving frame of reference which is responsible for the existence of that force.

[9] In literature one can find other terms for this kind of force, as *pseudo-force* or *fictitious force.*

This force might be considered fictitious only for outside observers, who are firmly standing on the Earth and build up their reasoning without knowing that the particle is in the accelerating streetcar, which leads them to Eq. (A1_1).

For the inside observer that force – being often paradoxically called *fictitious* – is almost *real* since it could be physically felt and experimentally measured. So the currently used term, i.e. *fictitious*, might appear rather misleading to observers living in a non-inertial frame of reference, i.e. in the accelerating street car.

Such contradictory terminology appears frequently. For example, Dare A. Wells in [6] states that *we shall, throughout the book, refer to the product (mass)×(acceleration) as an 'inertial force'*, while for C. Lancozs in [2] the inertial force is $-ma$. These two authors, as well as many others, are using the same term, i.e. inertial force, for two forces of the same magnitude differing, however, by a plus or minus sign.

Double integration of Eqs. (A1_6) with respect time, gives the velocity and coordinate distribution as functions of time with respect the accelerating street car – the distributions seen by an inside observer.

$$v_{x'} = v_{x'0} + \frac{Ft}{m}\cos\alpha - at, \quad v_{y'} = v_{y'0} + \frac{Ft}{m}\sin\alpha - gt. \tag{A1_7a}$$

$$x' = v_{x'0}t + \frac{Ft^2}{2m}\cos\alpha - \frac{1}{2}at^2, \quad y' = v_{y'0}t + \frac{Ft^2}{2m}\sin\alpha - \frac{1}{2}gt^2. \tag{A1_7b}$$

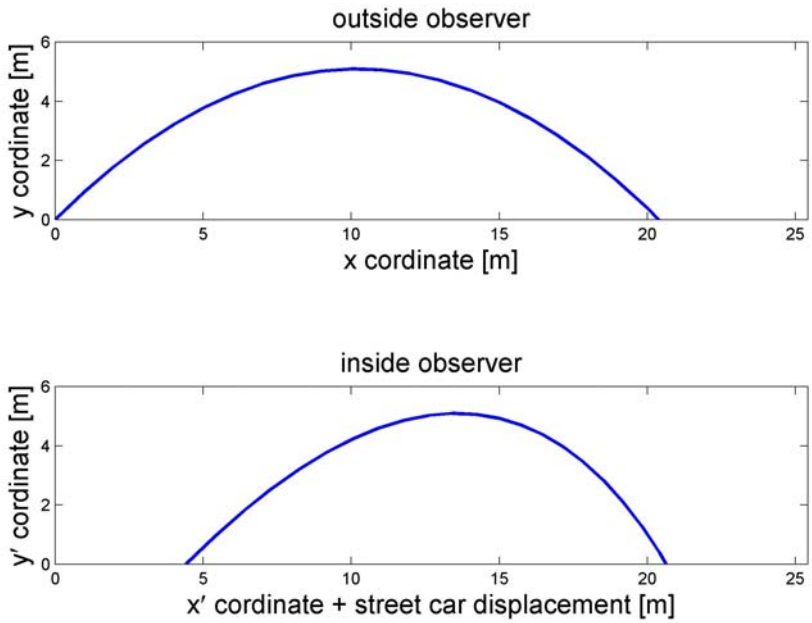


Fig. D21. The trajectory of a particle as viewed by two observers

In the upper subplot of Fig. D21 there is a trajectory (see Eq. (A1_2b)) of the particle, fired from the accelerating street car with prescribed initial velocities, as seen by an outside observer. In order to compare the results with those of a simple oblique shot case, the value of the rocket force F was temporarily set to zero. The data in the lower subplot of Fig. D21 show the trajectory as seen by the inside observer. The curve, plotted according to Eq.

(A1_7b), is intentionally shifted to the right by the distance the car had traveled during the period of the time being considered so that one can see that the coordinates of the 'hit' point, are identical for both observers. Of course, the time that elapses before the 'time to hit' is identical for both observers too.

Extended Example A2 – equatorial express

Imagine a train running on the track with a constant speed around the equator of the Earth in the opposite direction to the Earth's rotation, i.e. clockwise. A simplified sketch is in Fig. D22. The train, represented by a sleeve, is denoted by the number 3. The equatorial track, firmly connected to the Earth, depicted as a part of the circle, is denoted by the number 2. The number 1 is a fixed point in the Universe – an inertial frame of reference.

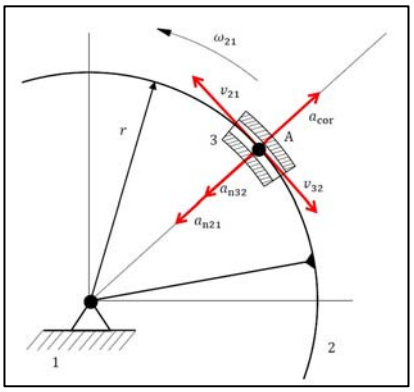


Fig. D22. Velocities and accelerations

Given: The radius of the Earth is r, the speed of the train with respect to the track is $v_{32} = |\vec{v}_{32}|$, the angular velocity of the Earth for its counterclockwise rotation is defined by a vector of angular velocity $\vec{\omega}_{21}$ pointing up vertically out of the picture plane. Its scalar magnitude is $\omega_{21} = |\vec{\omega}_{21}|$. The corresponding surface speed of a point on the equator, say A, is $v_{21} = r\omega_{21}$. The situation is schematically depicted in Fig. D22, where the directions of the speeds are indicated as well. The mass of the train m is concentrated at the point A.

Determine: Evaluate the force reaction between the track and the train – i.e. the actual weight of the train – as a function of its relative speed with respect to the Earth.

Simplifying assumptions

Since the relative differences in the actual weight of the train, depending on its location and speed, are of the order of a fraction of one percent, the precise simplifying assumptions have to be carefully listed.

- The Earth is assumed to be a perfect sphere with a constant radius; $r = 6378\,\text{km}$.
- The gravitational acceleration on the Earth's surface is constant and equal to $g = 9.81\,\text{m/s}^2$.
- The Earth's angular velocity is approximated by $\omega = \dfrac{2\pi}{24 \times 60 \times 60}\ 1/\text{s}$.
- The orbit of Earth around the Sun is disregarded.
- The actual weight of an object depends on its location on the Earth. When weighing the object of mass m by a spring balance on the pole, the balance shows the value of mg. Using the same spring balance and the same object at the equator the weight is diminished (due to the Earth rotation) by the value of the centrifugal force, i.e. by

$mr\omega^2$. The relative difference of these two values is 0.0034. The same object, measured at the equator would be lighter. Its actual weight would be $0.9966\,mg$.

- The actual weight also depends on its velocity with respect to the surface of the Earth.
- Resistance and friction phenomena are not considered.
- In Newtonian mechanics, the mass of an object is considered independent of its speed.

Kinematics

The train moves with respect to the Earth, which simultaneously rotates underneath the train. In this particular case, the decomposition of motions could be expressed by a symbolic notation in the form

31 = 32 + 21. (A2_1)

This means that

- the absolute motion of the train 3 with respect to inertial frame 1 is composed of
- the relative motion of the train 3 with respect the track 2 plus
- the motion of the track 2 with respect to the inertial frame 1.

For velocities, we can write

$$\vec{v}_{31} = \vec{v}_{32} + \vec{v}_{21}\,. \qquad \text{(A2\_2)}$$

The speed (the magnitude of velocity) of the train $v_{32} = |\vec{v}_{32}|$ is constant and known. The speed of the surface point just below the train is constant as well

$$v_{21} = |\vec{v}_{21}| = \omega_{21} r\,, \text{ where } \omega_{21} = |\vec{\omega}_{21}|\,. \qquad \text{(A2\_3)}$$

The resulting acceleration [9] with respect to the inertial frame is expressed by

$$\vec{a}_{31} = \vec{a}_{32} + \vec{a}_{21} + \vec{a}_{\text{cor}}\,. \qquad \text{(A2\_4)}$$

Generally, the acceleration vectors $\vec{a}_{32}$ and $\vec{a}_{21}$ have both tangential and normal components. In our case, both the rotation of the Earth and the velocity of the train with respect to the track are considered constant, so the tangential components of these accelerations are identically equal to zero, i.e.

$$a_{32\text{t}} = a_{21\text{t}} = 0\,. \qquad \text{(A2\_5)}$$

What remains are normal (centripetal) components of accelerations. Their magnitudes are

$$a_{32\text{n}} = v_{32}^2 / r \text{ and } a_{21\text{n}} = v_{21}^2 / r\,, \qquad \text{(A2\_6)}$$

while their directions are indicated by arrows in Fig. D22.

The Coriolis acceleration is defined as a vector product of the angular velocity of rotation and the relative velocity. In our case

$$\vec{a}_{\text{cor}} = 2\vec{\omega}_{21} \times \vec{v}_{32} \,. \qquad \text{(A2\_7)}$$

Since the vectors $\vec{\omega}_{21}$ and $\vec{v}_{32}$ are perpendicular, the magnitude of the resulting vector $|\vec{a}_{cor}|$ is

$$a_{\text{cor}} = 2\omega_{21} v_{32} = 2 v_{21} v_{32} / r \,. \qquad \text{(A2\_8)}$$

For the accepted clockwise train rotation the vector $\vec{a}_{\text{cor}}$ points out of the centre of rotation as indicated in Fig. D22. In this case, it is in the opposite direction with respect to directions of normal acceleration vectors $a_{32\text{n}}$ and $a_{21\text{n}}$. The vector $\vec{a}_{\text{cor}}$ would have the same magnitude but an opposite direction if the train rotation were considered counterclockwise. So the magnitude of the resulting radial acceleration with respect to the inertial frame is

$$a_{31} = m v_{32}^2 / r + m v_{21}^2 / r - 2 m v_{32} v_{21} / r \,. \qquad \text{(A2\_9)}$$

Evaluating Eq. (A2_9), for the train speed v_{32} varying from zero to $2v_{21}$, we obtain the resulting normal acceleration of the train as a function of its speed. Its normalized value, related to the gravitational acceleration, i.e. a_{31}/g, as a function of the normalized relative speed $|v_{32}/v_{21}|$, is plotted on the left-hand side of Fig. D24.

Dynamics

The forces acting upon the mass particle, act along a single line connecting the center of rotation and the point A, at which the mass particle, representing the train, is located. See Fig. D23. Hence, in the subsequent analysis, it suffices to express the equilibrium of forces in the scalar form.

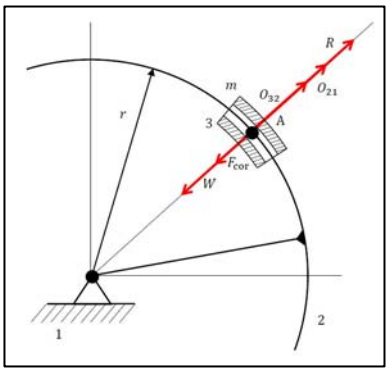


Fig. D23. Forces

Sitting on the train, we write the equations of motion using the d'Alembert approach [2] requiring us to consider the apparent inertia forces. Each apparent inertial force is defined as a product of the mass and the appropriate negative accelerations as follows

$O_{32} = m v_{32}^2 / r$ – centrifugal force due to the relative motion 32 with centripetal acceleration $a_{32\text{n}}$,
$O_{21} = m v_{21}^2 / r$ – centrifugal force due to the carrier rotation 21 with centripetal acceleration $a_{21\text{n}}$,
$F_{\text{cor}} = 2 m v_{32} v_{21} / r$ – Coriolis force due to the carrier rotation ω_{21} and the relative speed v_{32}.

Furthermore, due to Newton's gravitational law, there are the reaction force R, between the track and the train, and the train's actual weight W, which also has to be taken into account. Forces and their directions are shown in Fig. D23.

The equation of motion of the train (considered as a particle) has a form of 'dynamic' equilibrium

$$R + O_{32} + O_{21} - W - F_{cor} = 0 . \qquad \text{(A2\_10)}$$

So the reaction force is

$$R = W + F_C - O_{32} - O_{21} = mg + 2mv_{32}v_{21}/r - mv_{32}^2/r - mv_{21}^2/r . \qquad \text{(A2\_11)}$$

The reaction force between the track and the trains corresponds to the train's actual weight. Its relative value, i.e. R/mg, as a function of the absolute value of the relative speed $|v_{32}/v_{21}|$, is plotted in the right-hand side of Fig. D24.

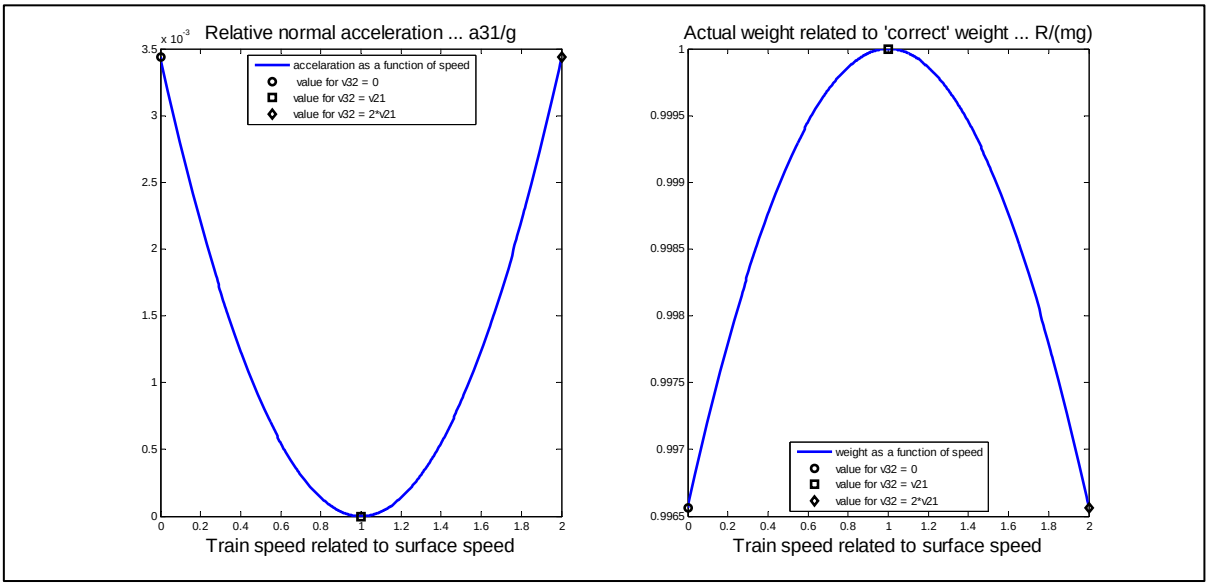


Fig. D24. The normal acceleration and the actual weight of a train circling the equator clockwise as functions of relative speed.

Now, a few singular cases are discussed in detail.

Case 1 – stationary train at the pole, i.e. $v_{32} = v_{21} = 0$.

If a stationary object (train) is weighed at the Earth's pole using a spring balance we would get the value of its weight (the force of gravity) which is influenced neither by the Earth's rotation nor by the object's speed. Under these conditions the reaction between the object and the Earth is $R = mg$. This location might serve for the definition of the value of the 'correct' weight.

Case 2 – stationary train at the equator, i.e. $v_{32} = 0$.

The train is stationary with respect to the Earth, so $v_{32} = 0$. In this case, the 'correct' weight of the train is diminished by the centrifugal force $O_{21} = mv_{21}^2/r$ due to the rotation of the Earth. Thus, the actual value of the weight is $R = mg - mr\omega^2$.

Case 3 – the train circling the equator clockwise with $\vec{v}_{32} = -\vec{v}_{21}$.
The train runs on the track around the equator in the opposite direction to the Earth's rotation. For velocity vectors we have $\vec{v}_{32} = -\vec{v}_{21}$. Their magnitudes, called speeds, are identical, i.e. $v = v_{32} = v_{21}$. The resulting velocity of the train $\vec{v}_{31}$, with respect to fixed stars, is identically equal to zero, which directly comes from Eq. (A2_2).

The resulting acceleration $\vec{a}_{31}$, with respect to fixed stars, according to the rearranged Eq. (A2_9), is

$$a_{31} = mv^2 / r + mv^2 / r - 2mvv / r = 0$$

and is equal to zero as well.

So, the **outside observer**, firmly attached to the fixed stars, i.e. to the inertial frame of reference, sees the train as a stationary object with zero velocity $\vec{v}_{31}$ and with zero acceleration $\vec{a}_{31}$.

In the inertial frame of reference, the train is stationary and is subjected to no acceleration. As a result, there are no inertial forces and there is no need to talk about dynamics. The only forces acting on the train are the reaction force R between the track and the train and its weight W resulting from Newton's gravitational law. Applying the static conditions of equilibrium leads to $R = mg$. So, in this case, the actual weight of the train, circling the equator clockwise with $\vec{v}_{32} = -\vec{v}_{21}$, is the same as that measured on the pole.

What about the **inside observer**, travelling on the equatorial train? Of course, he/she uses the same equation (A2_4) as far as the acceleration is concerned, but his/her attention is concentrated on the right-hand side of the equation. The resulting zero on the left-hand side is composed of three non-zero components. And according to d'Alembert's principle, each acceleration component is complemented by a corresponding apparent (fictitious) inertial force, defined as a product of mass and negative acceleration, in agreement with Eq. (A2_11). So in this case, we have

$$R = W + F_{\text{cor}} - O_{32} - O_{21} = mg + 2mv^2 / r - mv^2 / r - mv^2 / r .$$

The Coriolis force and the two centrifugal forces cancel themselves out and thus the reaction force is

$$R = mg .$$

We have obtained the same result both for inertial and non-inertial observers independently of the method of observation. This is rewarding.

Finally, what is the speed v_{32} of the train with respect to the Earth satisfying the conditions of Case 3? As far as the magnitudes of vectors are concerned we have

$$v = v_{32} = v_{21} = r\omega_{21} = 6378000\frac{2\pi}{24\times 60\times 60} = 463.8 \text{ m/s} = 1670 \text{ km/h}.$$

This is a high value, but not excessively so. At the expense of this relatively high relative speed, needed for satisfying Case 3 conditions, we get the same weight as that measured on the pole. With respect to the weight of the stationary train (Case 2, $v_{32} = 0$), the train traveling clockwise, with $\vec{v}_{32} = -\vec{v}_{21}$, is heavier by 0.34 %.

Case 4 – train circling the equator clockwise with $\vec{v}_{32} = -2\vec{v}_{21}$.
Both the acceleration and the reaction force are the same as in the Case 2. See Fig. D24. The detailed analysis of this case is left to the reader.

Example – stationary merry-go-round problem

A chain merry-go-round turns with a constant velocity ω. The seat plus the person sitting on it, having the mass m, are considered as a particle. The massless rope of the length l is attached to the frame of the merry-go-round at the distance r from the rotation axis. The free body diagram is in Fig. D25.

Given: r, l, ω, m
Determine: α

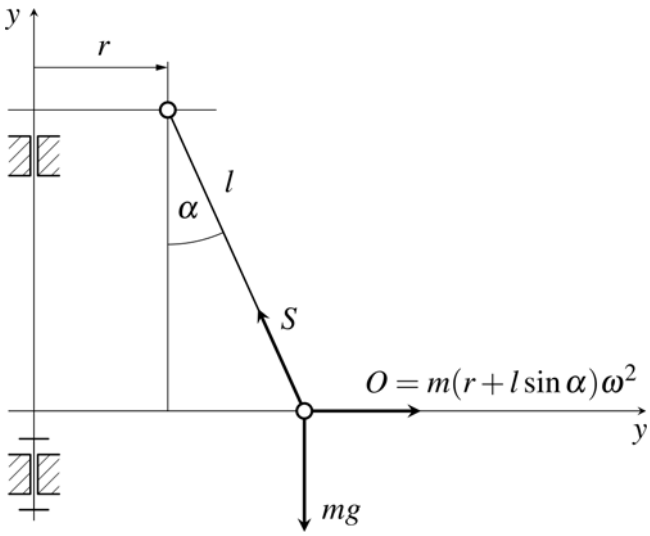


Fig. D25. Merry-go-round

Using the d'Alembert approach, the equations of motion are

$$O - S\sin\alpha = 0,$$
$$S\cos\alpha - mg = 0.$$

where

the rope force is denoted S and the centrifugal force is $O = m(r + l\sin\alpha)\omega^2$. Rearranging we get

$$mr\omega^2 + ml\omega^2\sin\alpha - \frac{mg}{\cos\alpha}\sin\alpha = 0,$$
$$r\omega^2 + l\omega^2\frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}} - g\tan\alpha = 0.$$

To simplify the solution we introduce a new auxiliary variable, say $x = \tan\alpha$, and then

$$r\omega^2 + l\omega^2 \frac{x}{\sqrt{1+x^2}} - gx = 0.$$

For given values of r, l it is easier to evaluate angular frequency ω as a function of x, i.e. as a function of angle α. So the inverse formula, i.e. $\omega = \sqrt{\dfrac{gx}{r + \dfrac{lx}{\sqrt{1+x^2}}}}$, is programmed, instead of the required function $x = f(\omega)$.

For details see the program `D06_merry_go_round_stationary_solution` and its graphical output shown in Fig. D26.

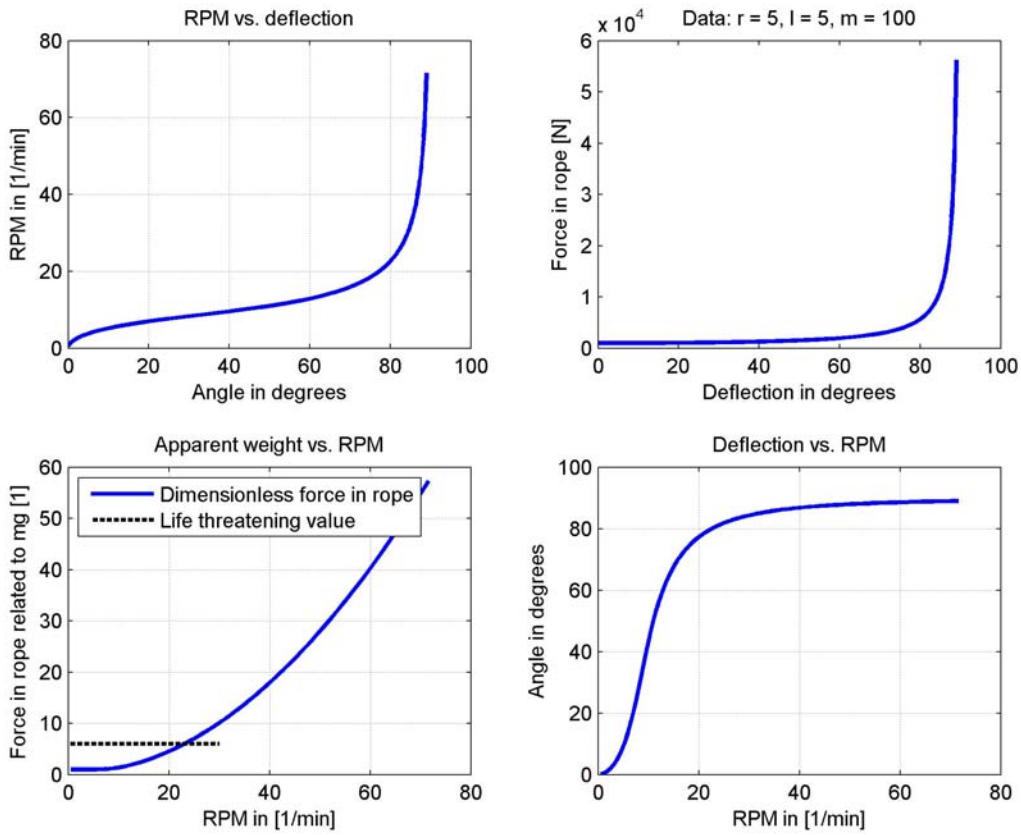


Fig. D26. Matlab output. Results for $r = 5$, $l = 5$, $m = 100$

```
% D06_merry_go_round_stationary_solution
% original file name is merry_go_round_c5
clear
% alpha ... angular deflection
% x ... tan(alpha)
% r ... radius
% n ... number of revolutions per minute (RPM)
% om = pi*n/30 ... angular velocity
% l ... length of rope
% g ... gravitaional acceleration
% m ... mass
%
% alpha_r ... angle in radians
% alpha_d ... angle in degrees
```

```
r = 5;          %[m]
l = 5;          % [m]
g = 9.81;       % [m/s^2]
m = 100;         % kg

% range of possible angular deflections
alfa_d = 0.1:0.1:89;            ... in degrees
alfa_r = pi*alfa_d/180;     ... in radians
R = r + l*sin(alfa_r);
om2 = g*tan(alfa_r)./R;
om = sqrt(om2);
% x = tan(alfa_r);
% om = sqrt(g*x./(r + l*x./sqrt(1 + x.*x)));
n = om*30/pi;           % ... RPM from angular velocity
S = m*g./cos(alfa_r);   % ... force in rope
S0 = m*g;               % ... initial weight
S_rel = S/S0;           % ... force related to original weight

figure(1)
xx = [90 90]; yy = [0 80];
subplot(2,2,1); plot(alfa_d,n, 'linewidth', 2); grid;
xlabel('Angle in degrees'); ylabel('RPM in [1/min]')
title('RPM vs. deflection')
txt = ['Data: ','r = ' num2str(r), ', l = ' num2str(l) ', m = ' num2str(m)];
subplot(2,2,2); plot(alfa_d,S, 'linewidth', 2); % grid
title(txt)
xlabel('Deflection in degrees'); ylabel('Force in rope [N]')
xxg =[0 30]; yyg = [6 6];
subplot(2,2,3); plot(n,S_rel,'b-', xxg,yyg,'k--', 'linewidth', 2); grid
title('Apparent weight vs. RPM')
ylabel('Force in rope related to mg [1]'); xlabel('RPM in [1/min]')
legend('Dimensionless force in rope', 'Life threatening value', 2)
subplot(2,2,4); plot(n,alfa_d, 'linewidth', 2); grid
ylabel('Angle in degrees'); xlabel('RPM in [1/min]')
title('Deflection vs. RPM')
print -djpeg -r300 merry_go_round_c5_fig_1

% simplified problem with r = 0
R = l*sin(alfa_r);
v_s = g*R.*tan(alfa_r);
om_s = sqrt(v_s./R);
n_s = om_s*30/pi;

figure(2)
plot(n,alfa_d,'b-', n_s,alfa_d,'b--', 'linewidth', 2)
title('Compare solutions');
ylabel('Angle in degrees'); xlabel('RPM in [1/min]')
legend('Full solution', 'Simplified case', 4)
axis([0 100 0 90])
print -djpeg -r300 merry_go_round_c5_fig_2
```

The dimensionless force in the rope, shown in the subplot (2,2,3) of Fig. D26. is the actual force related the force in the rope when the merry-go-round is in rest, i.e. to mg. Its value, depicted as a function of revolutions, shows how many times the tension in the rope is greater than its rest value and is clearly correlated to the riding comfort of a passenger. The values above five of six would be unacceptable. Recall the troubles of fighter pilots when subjected to high acceleration tests carried out on centrifugal machines.

Observing the value of the dimensionless tension force in the rope as a function of number of revolutions, one might estimate the safe range of operating conditions of the merry-go-round.

The problem is simplified if $r = 0$ and $R = l \sin\alpha$. See Fig. D27. Then, the equation of motion are

$$mR\omega^2 = S\sin\alpha\,,$$
$$mg = S\cos\alpha\,,$$
$$\Rightarrow \tan\alpha = \frac{R\omega^2}{g}\,.$$

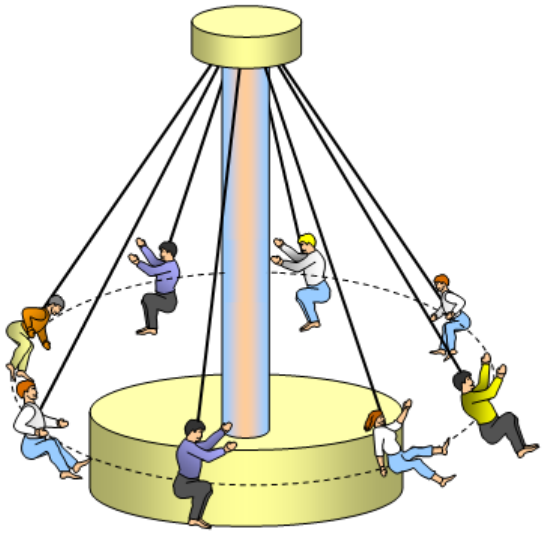


Fig. D27. Merry-go-round

Since $\omega = v/R$ we get the required velocity to attain the inclination α in the form

$$v^2 = gR\tan\alpha\,.$$

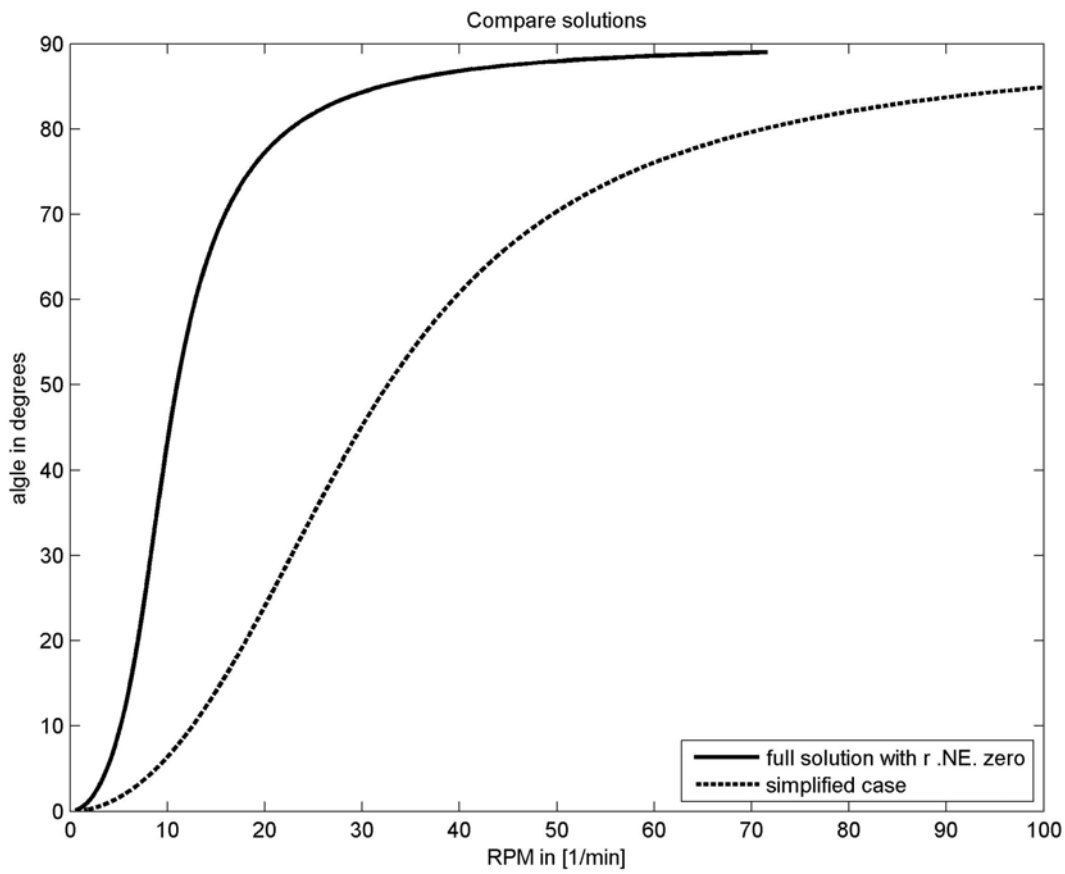


Fig. D28. Merry-go-round – two solutions

From the engineering point of view the mathematically simple case with $r = 0$, would be difficult to realize. The comparison of two discussed cases is in Fig. D28.

Example – a collar sliding along the rotating rod

Given: A collar, considered as a particle of mass m, could freely move along a rod, being perpendicularly welded to a shaft, which rotates by a constant angular velocity $\vec{\omega}$. Its magnitude is $|\vec{\omega}| = \omega = \omega_{21}$. See Fig. D29. The gravity and friction effects are neglected. The immediate position of the particle is denoted by x coordinate. The initial conditions: For $t = 0$ the position and velocity are $x = x_0, \dot{x} = v_0$.

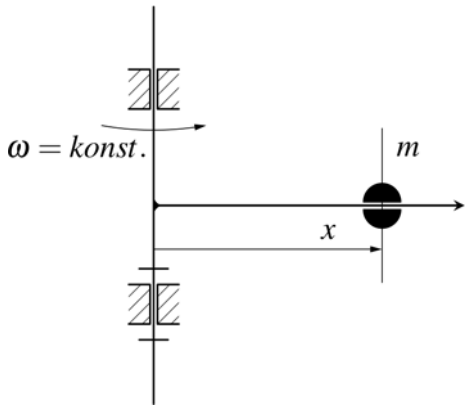


Fig. D29. A collar sliding along the rotating rod

Determine: The equations of motion, solve them and express the distance x as a function of time and of the rotation angle, say α.

If the particle is labeled by the number 3 and the rotating shaft by 2, and the fixed frame by 1, then the motion of the particle may be schematically described as

31 = 32 +21,

meaning that the resulting motion of the particle with respect to the frame (31) is composed of the relative motion of the particle with respect to the rotating rod (32) plus the carrier motion of the shaft (21) with respect to the frame 1.

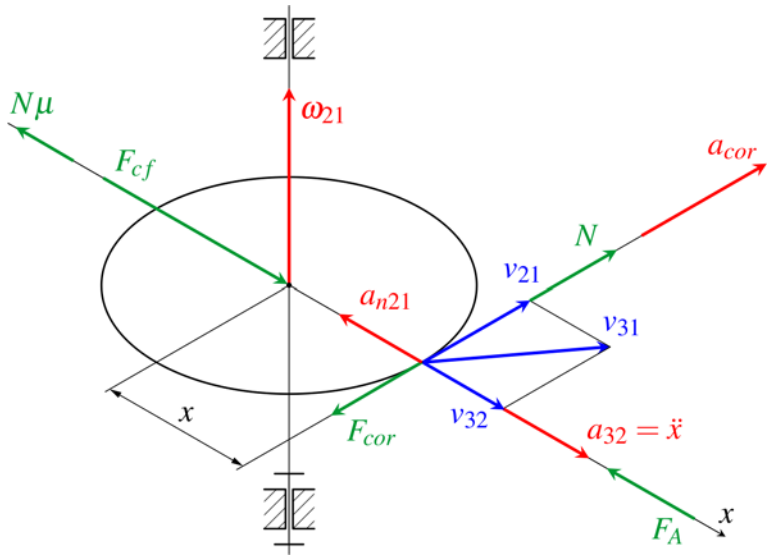


Fig. D30. Kinematics and dynamics

See Fig. D30. The resulting velocity of the particle is expressed by

$$\vec{v}_{31} = \vec{v}_{32} + \vec{v}_{21},$$

where the corresponding scalar values are

$$v_{32} = \dot{x}, \quad v_{21} = x\omega.$$

Since the carrier motion is of rotary nature the resulting acceleration contains the Coriolis acceleration term, so

$$\vec{a}_{31} = \vec{a}_{32} + \vec{a}_{21} + \vec{a}_{\mathrm{Cor}},$$

where the relative acceleration is

$$a_{32} = \ddot{x}.$$

Since the angular velocity $\omega_{21} = \mathrm{const}$, then the tangent component of the carrier acceleration $\vec{a}_{\mathrm{t}21} = 0$. The carrier acceleration $\vec{a}_{21}$ is given by its normal component $\vec{a}_{\mathrm{n}21}$ only since $\omega_{21} = \mathrm{const}$ and thus $\varepsilon_{21} = 0$.

$$|\vec{a}_{21}| = |\vec{a}_{\mathrm{n}21}| = x\omega^2 \quad .$$

Finally, the Coriolis acceleration is

$$\vec{a}_{\mathrm{Cor}} = 2\vec{\omega}_{21} \times \vec{v}_{32}.$$

Since the vectors $\vec{\omega}_{21}, \vec{v}_{32}$ are perpendicular, the magnitude of Coliolis acceleration is simply

$$a_{\mathrm{Cor}} = |\vec{a}_{\mathrm{Cor}}| = 2\omega_{21}v_{32} = 2\omega_{21}\dot{x}.$$

For dynamics of non-inertial systems, D'Alembert introduces apparent inertia forces being multiples of mass and corresponding negative accelerations. In this case we have d'Alembert apparent inertia force $F_{\mathrm{A}} = m\ddot{x}$, centrifugal force is $F_{\mathrm{cf}} = mx\omega_{21}^2$ and Coriolis force is $F_{\mathrm{Cor}} = ma_{\mathrm{Cor}}$. Directions of velocities, accelerations and forces are indicated in Fig. D30 by arrows. Since there are no forces acting within the direction of the rotation axis, two scalar equations of motion for the considered particle in 3D space are needed only.

$$m\ddot{x} - mx\omega_{21}^2 = 0,$$
$$N - 2m\omega_{21}\dot{x} = 0,$$

where N is the normal reaction between the particle and the rotating rod.

Note: If the friction is taken into account, then the friction force Nf, where f is the friction coefficient, will act along the x axis as it is indicated in Fig. D30, where $N = 2m\omega_{21}\dot{x}$. Then, the first equation of motion would have the form $m\ddot{x} + 2mf\omega_{21}\dot{x} - mx\omega_{21}^2 = 0$.

Dividing the first equation of motion by m and simplifying the notation by $\omega = \omega_{21}$ we get

$$\ddot{x} - \omega^2 x = 0.$$

The equation could be solved by means of so-called characteristic equation

$$\lambda^2 - \omega^2 = 0, \quad \Rightarrow \quad \lambda_{1,2} = \pm\omega$$

and then the solution is expressed in the form

$$x = C_1 \exp(\lambda_1 t) + C_2 \exp(\lambda_2 t) = C_1 \mathrm{e}^{\omega t} + C_2 \mathrm{e}^{-\omega t}.$$

The unknown integration constants are obtained from two initial conditions.
For $t = 0, \quad x = x_0$, so

$$x_0 = C_1 + C_2. \qquad \text{(a)}$$

For $t = 0, \quad \dot{x} = v_0$.

The second condition for finding unknown integration constants requires evaluating the derivative of the assumed solution $x = C_1 \mathrm{e}^{\omega t} + C_2 \mathrm{e}^{-\omega t}$, which gives

$$\dot{x} = C_1 \omega \mathrm{e}^{\omega t} - C_2 \omega \mathrm{e}^{-\omega t}.$$

Substituting the initial conditions into the derivative of x we get

$$v_0 = C_1 \omega - C_2 \omega. \qquad \text{(b)}$$

From (a) and (b) we obtain

$$C_1 = \frac{x_0 \omega + v_0}{2\omega}, \quad C_2 = \frac{x_0 \omega - v_0}{2\omega}.$$

So, the displacement of the particle, sliding along the frictionless rotating rod, as a function of time, is

$$x = \frac{x_0 \omega + v_0}{2\omega} \mathrm{e}^{\omega t} + \frac{x_0 \omega - v_0}{2\omega} \mathrm{e}^{-\omega t}.$$

The program `D07_projectile_rotating_barrel` generates Fig. D31 depicting the displacement of the projectile as a function of time and also as a function of the angle of rotation $\alpha = \omega t$. It is assumed that the initial conditions are: $t = 0, \quad \alpha = 0$.

See the program `projectile_rotating_barrel_edu_UL_04_kmi_odpudiva_sila`

```
% projectile_rotating_barrel_edu_UL_04_kmi_odpudiva_sila
clear
incr = 0.01;
x0 = 2; v0 = 3; om = 0.5;
t = 0:incr:7;
alpha = om*t;
C1 = (x0*om + v0)/(2*om);
C2 = (x0*om - v0)/(2*om);

x1 = C1*exp(om*t);
x2 = C2*exp(-om*t);

x = x1 + x2;
xdot = C1*om*exp(om*t) + C2*om*exp(om*t);
```

```
figure(1)
subplot(1,2,1)
plot(t,x,'k', 'linewidth', 2.5)
title('projectile in rotating barrel', 'fontsize', 16)
xlabel('time [s]','fontsize', 16); ylabel('displacement x [m]','fontsize', 16)

subplot(1,2,2)
polar(alpha,x,'ok'); grid
title('polar plot of displacement x', 'fontsize', 16)

print -djpeg -r300 fig_projectile_rotating_barrel
```

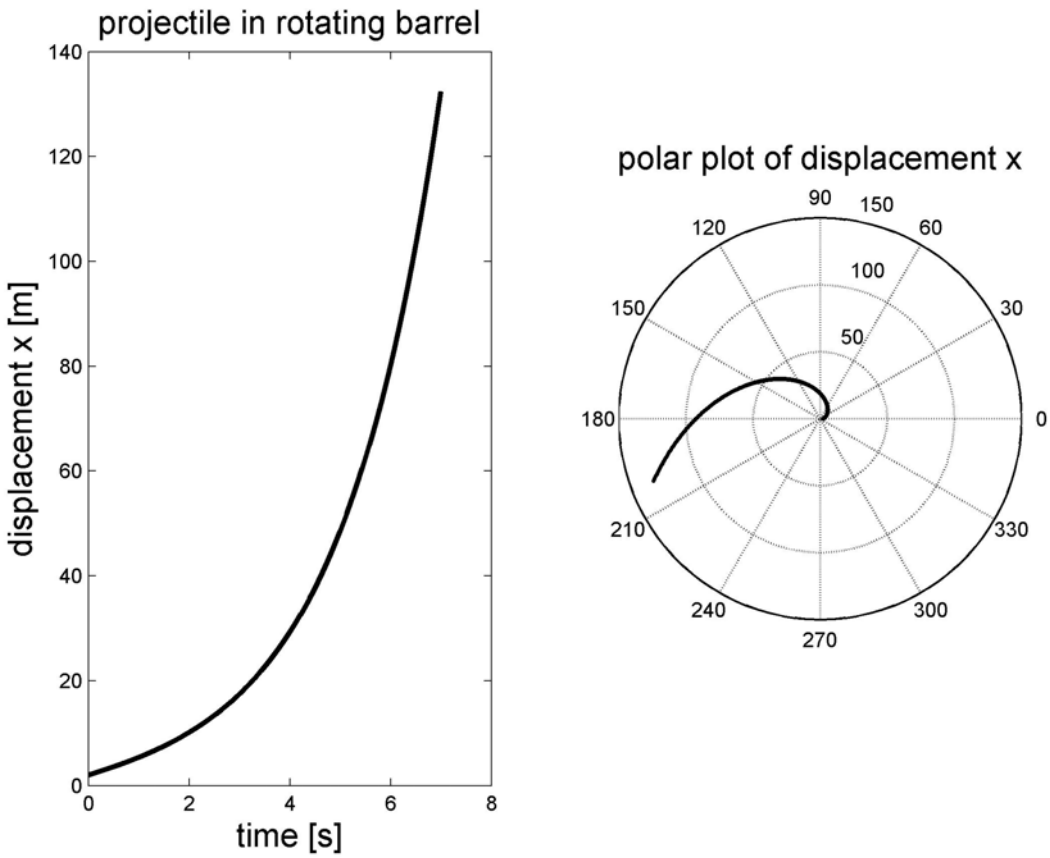


Fig. D31. Matlab output – displacements as functions of time and of angle

Example – motion of a particle in gravitational field

Newton gravitational law states that the attraction force between two bodies is directly proportional to the product of masses and indirectly proportional to the square of their distance. Let's apply the law to the motion of the Earth around the Sun, which is assumed to be in the origin of the coordinate system. See Fig. D32.

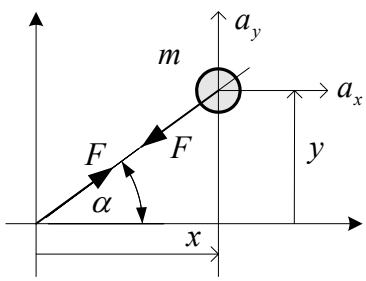


Fig. D32. Motion of a particle in gravitational field

$$F = \kappa \frac{mM}{r^2} = \frac{k}{r^2},$$

where $r = \sqrt{x^2 + y^2}$ is the immediate distance and the gravitational constant is $k = \kappa mM$.

Constants
$\kappa = 6.67\times10^{-11}\ \mathrm{m^3kg^{-1}s^{-2}}$, $M = 1.99\times10^{30}\ \mathrm{kg}$.

Initial conditions
a) Position
$x_0 = r = 1.5\times10^{11}\ \mathrm{m}$ … initial distance of the Earth from the Sun,
$y_0 = 0$.

b) Velocity of the Earth
$v_{x0} = 0$,
$v_{y0} = 2.9\times10^{4}\ \mathrm{m/s}$.

Equations of motion are

$$m\ddot{x} = -F\cos\alpha,$$
$$m\ddot{y} = -F\sin\alpha,$$

where $\cos\alpha = x/r = \dfrac{x}{\sqrt{x^2+y^2}}$; $\sin\alpha = y/r = \dfrac{y}{\sqrt{x^2+y^2}}$,

$$m\ddot{x} = -\frac{k\cos\alpha}{r^2} = -\frac{k}{x^2+y^2}\frac{x}{\sqrt{x^2+y^2}},$$
$$m\ddot{y} = -\frac{k\sin\alpha}{r^2} = -\frac{k}{x^2+y^2}\frac{y}{\sqrt{x^2+y^2}},$$

$$\ddot{x} = -\frac{k}{m}\frac{x}{\left(x^2+y^2\right)^{\frac{3}{2}}} = -\omega^2\frac{x}{\left(x^2+y^2\right)^{\frac{3}{2}}},$$
$$\ddot{y} = -\frac{k}{m}\frac{y}{\left(x^2+y^2\right)^{\frac{3}{2}}} = -\omega^2\frac{y}{\left(x^2+y^2\right)^{\frac{3}{2}}},$$

where we have introduced $\omega^2 = k/m = \kappa M$
$\omega^2 = 1.3273\mathrm{e}+020$.

ODE integrating procedures of Matlab require a system of the first order equations. A suitable substitution might be $\dot{x} = z;\quad \dot{y} = w$.

From it follows
$$\dot{z} = -\omega^2\frac{x}{\left(x^2+y^2\right)^{\frac{3}{2}}},\quad \dot{w} = -\omega^2\frac{y}{\left(x^2+y^2\right)^{\frac{3}{2}}}.$$

Rearrange, rename and relate to original notation.

$$
\begin{aligned}
&\dot{x} = z && \dot{x} = \dot{p}_1 \quad x = p_1 \quad \cdots \quad x, \\
&\dot{z} = -\omega^2 \frac{x}{\left(x^2 + y^2\right)^{\frac{3}{2}}} && \dot{z} = \dot{p}_2 \quad z = p_2 \quad \cdots \quad v_x, \\
&\dot{y} = w && \dot{y} = \dot{p}_3 \quad y = p_3 \quad \cdots \quad y, \\
&\dot{w} = -\omega^2 \frac{y}{\left(x^2 + y^2\right)^{\frac{3}{2}}} && \dot{w} = \dot{p}_4 \quad w = p_4 \quad \cdots \quad v_y.
\end{aligned}
$$

Assemble newly named variables and equations in an array fashion

$$
\begin{aligned}
&\dot{p}_1 = p_2, \\
&\dot{p}_2 = \frac{-\omega^2 p_1}{\left(p_1^2 + p_3^2\right)^{\frac{3}{2}}}, \\
&\dot{p}_3 = p_4, \\
&\dot{p}_4 = \frac{-\omega^2 p_3}{\left(p_1^2 + p_3^2\right)^{\frac{3}{2}}}.
\end{aligned}
$$

Rename again

$$
\begin{aligned}
&\dot{y}_1 = y_2, \\
&\dot{y}_2 = \frac{-\omega^2 y_1}{\left(y_1^2 + y_3^2\right)^{\frac{3}{2}}}, \\
&\dot{y}_3 = y_4, \\
&\dot{y}_4 = \frac{-\omega^2 y_3}{\left(y_1^2 + y_3^2\right)^{\frac{3}{2}}}.
\end{aligned}
$$

Matlab implementation of equations of motion is provided by the function `central.m`.

```
function dydt = central(t,y)
Omega2 = 1.3273e+020;

dydt = [y(2); -Omega2*y(1)/sqrt((y(1)^2 + y(3)^2))^3; ...
        y(4); -Omega2*y(3)/sqrt((y(1)^2 + y(3)^2))^3];

% end of function dydt = central(t,y)
```

The main program is

```
% test_central_c1
% numerical integration of equation of motion
% describing a motion of a particle in gravitational field
% force of gravitaton F = kappa*m*M/r^2 = k/r^2; Newton's law
% kappa is gravitational constant, mass of Sun is M, mass of Earth is m,
% r is the distance
kappa = 6.67e-11    % m^3 kg^-1 sec^-2;
M = 1.99e30         % kg
r = 1.5e11          % m
% k = kappa*m*M
% Omega2 = k/m = kappa*M;
Omega2 = kappa*M
Omega = sqrt(Omega2)
```

```
% m*ddotx = -F*cos(alpha)/r^2;
% m*ddoty = -F*sin(alpha)/r^2;
% r = sqrt(x^2 + y^2);
% cos(alpha) = x/r; sin(alpha) = y/r;
% Omega2 = k/m;
% xddot = -Omega2*x/((x^2 + y^2)^(3/2));
% yddot = -Omega2*y/((x^2 + y^2)^(3/2));

year = 365*24*3600
tspan = [0 year];   % timespan

% initial conditions
x0 = 150e9;         % m ... initial position of Earth, distance from Sun
vx0 = 0;
y0 = 0;
vy0 = 29600;        % m/s ... initial starting velocity of Earth
y0 = [x0 vx0 y0 vy0];  % initial conditions for ode function

[t,y] = ode23(@central,tspan,y0);

% Relation to original coordinates
% y(:,1) ... x
% y(:,2) ... vx
% y(:,3) ... y
% y(:,4) ... vy

figure(1)
subplot(2,2,1); plot(t,y(:,1)); title('x coordinate');  axis([0 year -2e11 2e11]);
xlabel('time in [s]')
subplot(2,2,2); plot(t,y(:,2)); title('vx velocity');   axis([0 year -4e4 4e4]);
xlabel('time in [s]')
subplot(2,2,3); plot(t,y(:,3)); title('y coordinate');  axis([0 year -2e11 2e11]);
xlabel('time in [s]')
subplot(2,2,4); plot(t,y(:,4)); title('vy velocity');   axis([0 year -4e4 4e4]);
xlabel('time in [s]')

figure(2)
xxS = 0;  yyS = 0;
xxE = x0; yyE = 0;
% plot circle
tt = 0:pi/64:2*pi;
xx = x0*cos(Omega*tt);
yy = x0*sin(Omega*tt);
plot(y(:,1),y(:,3), 'o-', xxS,yyS,'ok', xxE,yyE,'sk', xx,yy,'--', 'linewidth', 2.1)
xlabel('[m]', 'fontsize', 14); ylabel('[m]', 'fontsize', 14)
legend('computed elliptical orbit', 'Sun', 'Earth initial position', 'circle', 3)
axis('square'); axis([-1.6e11 1.6e11 -1.6e11 1.6e11])
text(-0.05e11,-0.1e11, 'Sun')

title('Earth elliptical orbit and a perfect cirle', 'fontsize', 18)
% orbital velocity
v_orbit = (y(:,2).^2 + y(:,4).^2).^(1/2);
figure(3)
subplot(2,1,1)
plot(y(:,2), y(:,4)); axis('square'); axis([-4e4 4e4 -4e4 4e4])
title('vx versus vy')
subplot(2,1,2)
plot(t,v_orbit)
title('abs. value of orbit velocity vs. time'); xlabel('time in [s]')
% end of test_central_c1
```

Computed and plotted data in Fig. D33 are for following inputs. Data are highly approximate.

Gravitational constant $\kappa = 6.67 \times 10^{-11}\ \mathrm{m^3 kg^{-1} s^{-2}}$,

Mass of the Sun $M = 1.99 \times 10^{30}\ \mathrm{kg}$.

Initial conditions

a) Position

$x_0 = r = 1.5 \times 10^{11}$ m … initial distance of the Earth from the Sun,

$y_0 = 0$.

b) Velocity of the Earth

$v_{x0} = 0$,

$v_{y0} = 2.96 \times 10^4$ m/s .

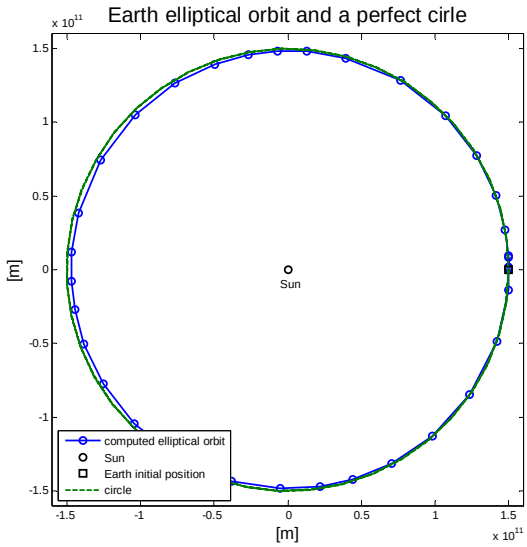


Fig. D33. Trajectory of the Earth around the Sun is almost circular

D6. Vibration

The subject is fully described in the chapter **Vibration** of the electronic publication prepared by Stejskal, V., Dehombreux P., Eiber, A., Gupta, R., Okrouhlík, M.: Mechanics with Matlab, pp. 301 – 461. Faculté Polytechnique de Mons, Belgium, April 2001, ISBN 2-9600226-2-9, see also http://www.geniemeca.fpms.ac.be.

D7. Moments of inertia and deviatoric moments

The moments of inertia and deviatoric moments are measures defining how the mass is distributed within a rigid body.

In statics, we have dealt with similar quantities called linear (static), and quadratic (also called the second) moments of area – we computed these quantities considering moments of planar or volumetric elements with respect to coordinate axes. Let's remind how we computed the quadratic moment of the cross-sectional area with respect to the *x*-axis. See Fig. D34.

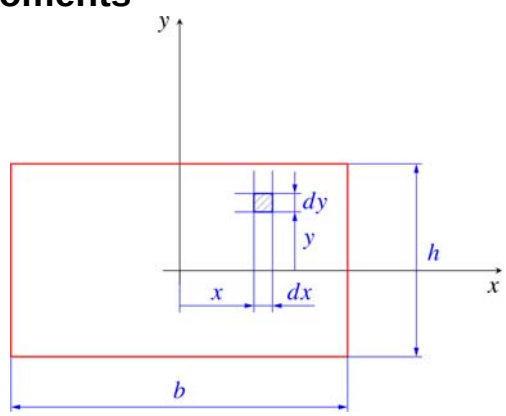


Fig. D34. Quadratic moment of a rectangular cross section

$$J_x = 4\int_0^{b/2}\int_0^{h/2} y^2\,\mathrm{d}x\,\mathrm{d}y = 4\int_0^{b/2}\left[\frac{y^3}{3}\right]_0^{h/2}\mathrm{d}x = 4\frac{1}{3}\int_0^{b/2}\frac{h^3}{8}\mathrm{d}x = \frac{4}{24}h^3\int_0^{b/2}\mathrm{d}x = \frac{4}{24}h^3[x]_0^{b/2} = \frac{4}{24}h^3\frac{b}{2} = \frac{1}{12}bh^3$$

In dynamics, instead of summing quadratic moments of elementary planar elements $\mathrm{d}x\,\mathrm{d}y$, we are evaluating sums of quadratic moments of masses belonging to mass elements, i.e. $\rho\, y^2\,\mathrm{d}x\,\mathrm{d}y$, where quantity ρ is the planar density measured in $[\mathrm{kg/m^2}]$.

Generally, the *moment of inertia* of a body of the mass m composed of n material particles m_i (or of all infinitesimal elements $\mathrm{d}m$) about an axis, say o, is defined by the relation

$$I_o = \sum R_i^2\, m_i = \int_m R^2\,\mathrm{d}m\,, \qquad \text{(D7\_1)}$$

where R^2 is the square of the shortest distance of each elementary mass $\mathrm{d}m$ from the considered axis. See Fig. D35.

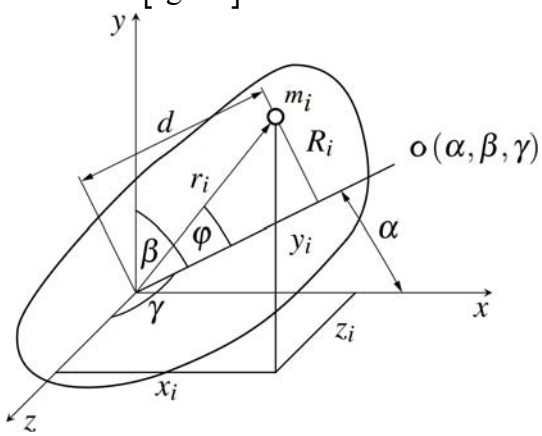


Fig. D35. Moments of inertia

Sometimes, another quantity named the *gyration radius* used. It is defined by the relation

$$r_\mathrm{g} = \sqrt{\frac{I_0}{m}}\,. \qquad \text{(D7\_2)}$$

Similarly, the moments of inertia about coordinate axes are defined by

$$\begin{aligned} I_x &= \int_m \left(y^2 + z^2\right)\mathrm{d}m, \\ I_y &= \int_m \left(z^2 + x^2\right)\mathrm{d}m, \\ I_z &= \int_m \left(x^2 + y^2\right)\mathrm{d}m. \end{aligned} \qquad \text{... (D7\_3)}$$

Sometimes, an alternative notation for moments of inertia is used, namely

$$I_{xx} = I_x,\, I_{yy} = I_y,\, I_{zz} = I_z\,. \qquad \text{(D7\_4)}$$

The moments of inertia with respect to coordinate planes are

$$I_{yz} = \int_m \left(x^2\right)\mathrm{d}m,\, I_{zx} = \int_m \left(y^2\right)\mathrm{d}m,\, I_{xy} = \int_m \left(z^2\right)\mathrm{d}m. \qquad \text{... (D7\_5)}$$

Observing the above relations, it is obvious that

$$I_x = I_{xy} + I_{zx},\, I_y = I_{yz} + I_{xy},\, I_z = I_{zx} + I_{yz}\,. \qquad \text{(D7\_6)}$$

The moment of inertia with respect to the origin of coordinate system is

$$I_{\mathrm{O}} = \int_m \left(x^2 + y^2 + z^2\right) dm = I_{xy} + I_{yz} + I_{zx} = \frac{I_x + I_y + I_z}{2}. \qquad \text{(D7\_7)}$$

The deviatoric moments are defined by

$$\begin{aligned} D_{xy} &= \int_m xy\, dm, \quad D_{yx} = \int_m yx\, dm = D_{xy}, \\ D_{yz} &= \int_m yz\, dm, \quad D_{zy} = \int_m zy\, dm = D_{yz}, \\ D_{zx} &= \int_m zx\, dm, \quad D_{xz} = \int_m xz\, dm = D_{zx}. \end{aligned} \qquad \text{... (D7\_8)}$$

The moments of inertia and deviatoric moments are often assembled into a single matrix known as the *inertia matrix*

$$\mathbf{I} = \begin{bmatrix} I_x & -D_{xy} & -D_{xz} \\ -D_{yx} & I_y & -D_{yz} \\ -D_{zx} & -D_{xy} & I_z \end{bmatrix} = \begin{bmatrix} I_{xx} & -D_{xy} & -D_{xz} \\ -D_{yx} & I_{yy} & -D_{yz} \\ -D_{zx} & -D_{xy} & I_{zz} \end{bmatrix}. \qquad \text{(D7\_9)}$$

Notice that the matrix is symmetric. The dimensions of elements of the inertia matrix are *mass* × *square of length*, i.e. $\left[\mathrm{kg\,m^2}\right]$.

Example – moments of inertia

Given: A cone having its apex in the origin of the coordinate system x, y is defined by its dimensions and the density ρ. See Fig. D36.

Determine: Moments of inertia with respect to coordinate axes x, y, z.

To simplify the computation the mass element is considered as a ring with the radius y, the height $\mathrm{d}y$ and the thickness $\mathrm{d}x$.

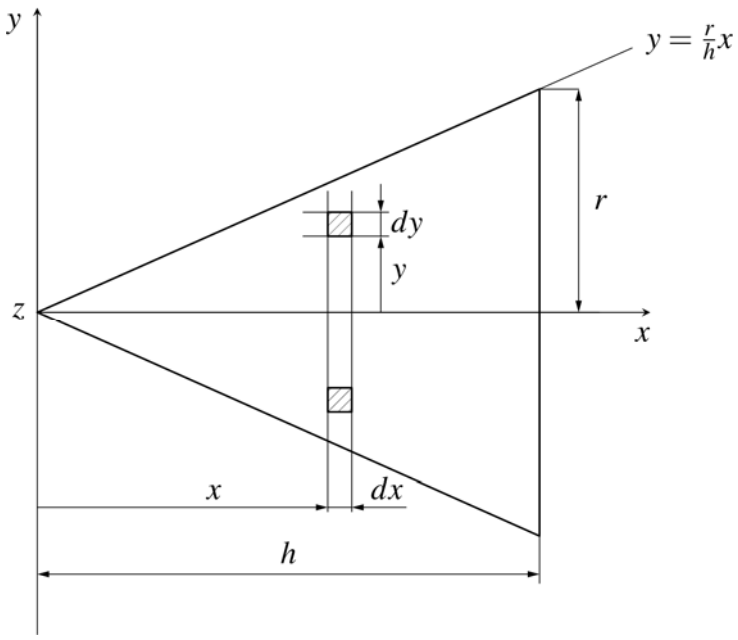


Fig. D36. Moment of inertia for a conus

Then, the moment of inertia about x axis is

$I_x = \int_m y^2 \, \mathrm{d}m$, where $dm = \rho 2\pi \, y \, \mathrm{d}x \, \mathrm{d}y$.

$$I_x = 2\pi\rho \int_0^h \int_0^{rx/h} y^3 \mathrm{d}x \, \mathrm{d}y = 2\pi\rho \int_0^h \left[\frac{y^4}{4} \right]_o^{rx/h} \mathrm{d}x = 2\pi\rho \int^h \frac{r^4 x^4 / h^4}{4} \mathrm{d}x = \frac{\pi\rho r^4}{2h^4} \int_0^h x^4 \mathrm{d}x = \frac{\pi\rho}{10} h r^4 .$$

Dimensional check: $\frac{\mathrm{kg}}{\mathrm{m}^3} \mathrm{m}^5 = \mathrm{kg}\,\mathrm{m}^2$.

In this example the density ρ was put in front of the integral sign since it is assumed that it is distributed homogeneously within the considered body.

In cases where the density is a function of spatial coordinates then it must stay behind the integral sign and be properly integrated with spatial coordinates describing the body's shape.

In engineering textbooks, one can find an alternative formula, i.e. $I_x = \frac{3}{10} m r^2$, where $m = \frac{1}{3} \pi \rho r^2 h$ is the mass of the cone. Check, that the formulas are identical.

Moment of inertia about y axis

$$I_y = \int r^2 \mathrm{d}m = \int \left(x^2 + z^2 \right) \mathrm{d}m = \int x^2 \mathrm{d}m + \int z^2 \mathrm{d}m = I_{yx} + I_{xy} .$$

We know that

$I_x = I_{xy} + I_{xz}$. And also that due to symmetry $I_{xy} = I_{xz}$, $\Rightarrow I_x = 2 I_{xz}$ and finally

$$I_{xz} = I_x / 2 = \frac{\pi\rho}{20} h r^4 .$$

Still, we have to determine

$$I_{yz} = \int x^2 \mathrm{d}m, \quad dm = \rho \pi y^2 dx, \quad y = rx / h.$$

Now, the mass element is taken as a circular plate of the radius y and the thickness $\mathrm{d}x$. So,

$$I_{yz} = \int_0^h \rho\pi \frac{r^2}{h^2} x^2 x^2 \mathrm{d}x = \frac{\pi\rho}{5} r^2 h^3 .$$

Finally, $I_y = I_{xy} + I_{yz} = \frac{\pi\rho}{20} h r^4 + \frac{\pi\rho}{5} h^3 r^2$, $I_z = I_y$.

Example – deviatoric moments

Given: A blade, depicted in Monge's projection with its dimensions according in Fig. D37, has the density ρ.
Determine: Deviatoric moments.

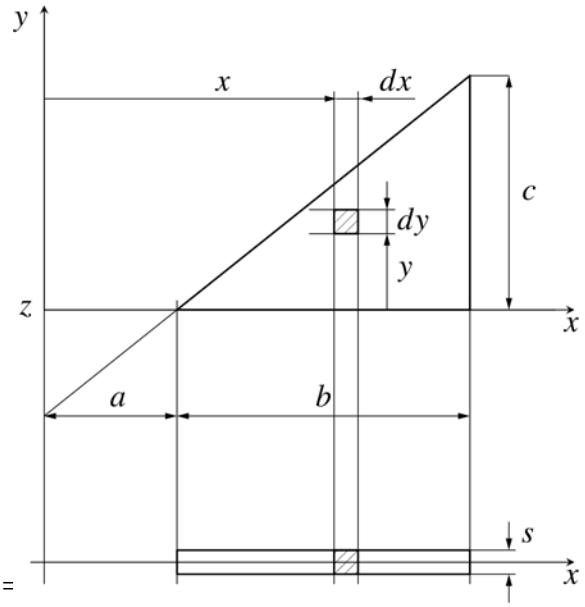


Fig. D37. Deviatoric moments

Due to symmetry we have $D_{yz} = 0, \quad D_{xz} = 0$.

$$D_{xy} = \int xy\,\mathrm{d}m\,,$$
$$dm = \rho s\,\mathrm{d}x\,\mathrm{d}y\,,$$
$$y = \frac{c}{b}x - \frac{c}{b}a = \frac{c}{b}(x-a)x\,.$$
$$D_{xy} = \int_0^{a+b}\int_0^{c(x-a)/b} \rho s\,xy\,\mathrm{d}x\,\mathrm{d}y = \rho s \int_a^{a+b} x\left[\frac{y^2}{2}\right]_0^{c(x-a)/b}\mathrm{d}x =$$
$$= \frac{\rho s c^2}{2b^2}\int_0^{a+b} x\left(x^2 - 2ax + a^2\right)dx =$$
$$= \frac{\rho s c^2}{2b^2}\int_0^{a+b}(x^3 - 2ax^2 + a^2x)\,\mathrm{d}x =$$
$$= \frac{\rho s c^2}{2b^2}\left[\frac{x^4}{4} - 2a\frac{x^3}{3} + a\frac{x^2}{2}\right]_a^{a+b} = \ \ldots\ = \frac{\rho s c^2}{2b^2}\left(\frac{b^4}{4} + \frac{ab^3}{3}\right).$$

Example – moment of inertia

Given: l, α and ρ – density per unit length. See Fig. D38.
Determine: I_x, D_{xy}.

$$I_x = \int y^2\mathrm{d}m, \quad D_{xy} = \int xy\,\mathrm{d}m\,.$$
$$dm = \rho\,\mathrm{d}s, \quad ds = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}\,.$$

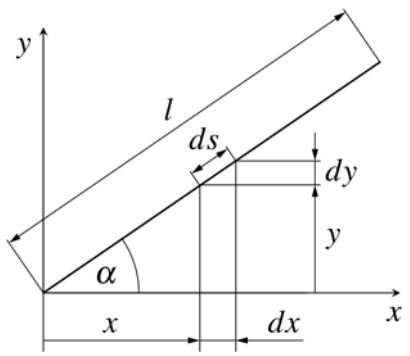


Fig. D38. Moment of inertia

$$y = kx, \quad k = \tan\alpha = \frac{\sin\alpha}{\cos\alpha}, \quad \sqrt{1+k^2} = \frac{1}{\cos\alpha}\,.$$
$$\mathrm{d}y = k\,\mathrm{d}x \Rightarrow ds = \mathrm{d}x\sqrt{1+k^2}\,.$$
$$I_x = \int y^2\mathrm{d}m = \int_0^{l\cos\alpha}\rho(kx)^2\sqrt{1+k^2}\,\mathrm{d}x = \rho k\sqrt{1+k^2}\int_0^{l\cos\alpha}x^2\mathrm{d}x = \frac{1}{3}\rho k^2\sqrt{1+k^2}\,l^3\cos^3\alpha =$$
$$= \frac{1}{3}\rho\frac{\sin^2\alpha}{\cos^2\alpha}\frac{1}{\cos\alpha}l^3\cos^3\alpha = \frac{1}{3}\rho l^3\sin^2\alpha\,.$$

Similarly $D_{xy} = \int xy\,\mathrm{d}m = \int_0^{l\cos\alpha} x(kx)\rho\sqrt{1+k^2}\,\mathrm{d}x = \ldots = \frac{1}{3}\rho l^3 \sin\alpha\cos\alpha$.

To understand the subject of mass distribution of rigid bodies we have presented a detailed procedure how moments of inertia are calculated. Usually, the formulas for moments of inertia as well as deviatoric moments are not computed from the scratch but are readily found in engineering textbooks instead. As an example, a few of frequently used formulas are presented below.

Moments of inertia of rigid bodies of mass m

body	axis	I
thin rod, length L	perpendicular axis through centre	$\frac{1}{12}mL^2$
thin ring, radius R	perpendicular axis through centre	mR^2
circular cylinder, radius R	axis of cylinder	$\frac{1}{2}mR^2$
thin disk, radius R	transverse axis through centre	$\frac{1}{4}mR^2$
sphere, radius R	any axis through centre	$\frac{2}{5}mR^2$
thin spherical shell, radius R	any axis through centre	$\frac{2}{3}mR^2$
thin rectangular plate, $a \times b$ ... axis through centre perpendicular to plate		$\frac{1}{12}m\left(a^2+b^2\right)$

D8. Dynamics of rigid bodies

It should be reminded that

- rigid bodies do not deform due to applied forces and moments,
- mass distribution within a body is characterized by
  - the location of the centre of mass,
  - the moments of inertia,
  - the deviatoric moments,
- usually, it is assumed that the density is distributed homogeneously within a body,
- a free body in 3D space has six degrees of freedom, thus six equations of motion are required (at least three of them have to be of moment nature),
- a free body in 2D space has three degrees of freedom, thus three equations of motion are required (at least one of them have to be of moment nature).

D8.1 Translatory motion

All the material points (particles) of the considered body have (in a given moment) the same trajectories, velocities, and accelerations. The angular velocities and angular accelerations are equal to zero.

Momentum	$\mathbf{p} = m\mathbf{v}$,	
Angular momentum about the centre of mass S	$\mathbf{L}_\mathrm{S} = \mathbf{0}$,	... (D8_1)
Angular momentum about a generic point O	$\mathbf{L}_\mathrm{O} = \mathbf{r}_\mathrm{S} \times m\mathbf{v}$,	
Kinetic energy	$E_\mathrm{k} = \frac{1}{2} m v^2$.	(D8_2)

Vector equations of motion about a generic point O are

$$m\mathbf{a} = \sum \mathbf{F}_i \,, \qquad \text{(D8\_4)}$$

$$\mathbf{r}_\mathrm{S} \times m\mathbf{a} = \sum \mathbf{M}_{i\mathrm{O}} \qquad \text{or about the centre of mass S} \qquad \mathbf{0} = \sum \mathbf{M}_{i\mathrm{S}} \,.$$

The above vector relations are generic. They are valid universally. For a body in 3D space we write six scalar equations instead. Equations of motion about a generic point O are

$$\begin{aligned} ma_x &= \sum F_{ix} \,, \\ ma_y &= \sum F_{iy} \,, \\ ma_z &= \sum F_{iz} \,, \end{aligned} \qquad \text{... (D8\_5)}$$

$$\begin{aligned} m(y_\mathrm{S} a_z - z_\mathrm{S} a_y) &= \sum M_{ix} & & 0 = \sum M_{i\mathrm{S}x} \,, \\ m(z_\mathrm{S} a_x - x_\mathrm{S} a_z) &= \sum M_{iy} & \text{or about the centre of mass S} \quad & 0 = \sum M_{i\mathrm{S}y} \,, \\ m(x_\mathrm{S} a_y - y_\mathrm{S} a_x) &= \sum M_{iz} & & 0 = \sum M_{i\mathrm{S}z} \,. \end{aligned} \qquad \text{(D8\_6)}$$

Hint – apparent inertia forces for a translatory motion

The apparent inertia forces and moments for a body subjected to a translatory motion, written with respect to the centre of mass S, are

$$\vec{D} = -m\vec{a}, \vec{M}_\mathrm{S}^\mathrm{D} = \vec{0} \,.$$

The vector $\vec{D}$ and the upper right index D stands for d'Alembert. See Fig. D39. Notice, that in accompanying pictures the shown vectors are denoted by oriented lines with arrows, showing the direction, and by letter labels having no above arrows. So the labels indicate just the magnitudes of particular vectors.

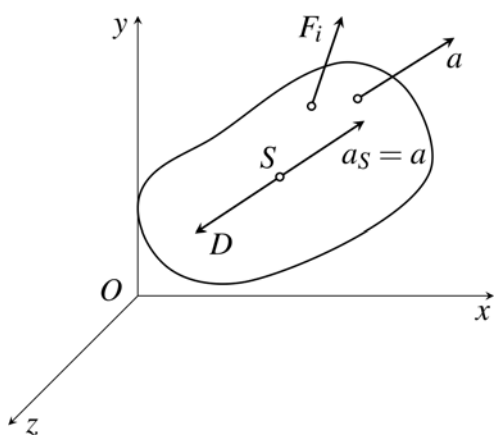


Fig. D39. Inertia forces – translation

Example – a skidding car on a slope

Given: Dimensions m, l, h, b, α, the coefficient of friction f and gravitational acceleration g. A car of the weight $Q = mg$, whose wheels are fully braked (no rotation), is skidding downwards the slope inclined by an angle α. The initial velocity of the car is v_0. See Fig. D40, where the free body diagram forces are indicated.

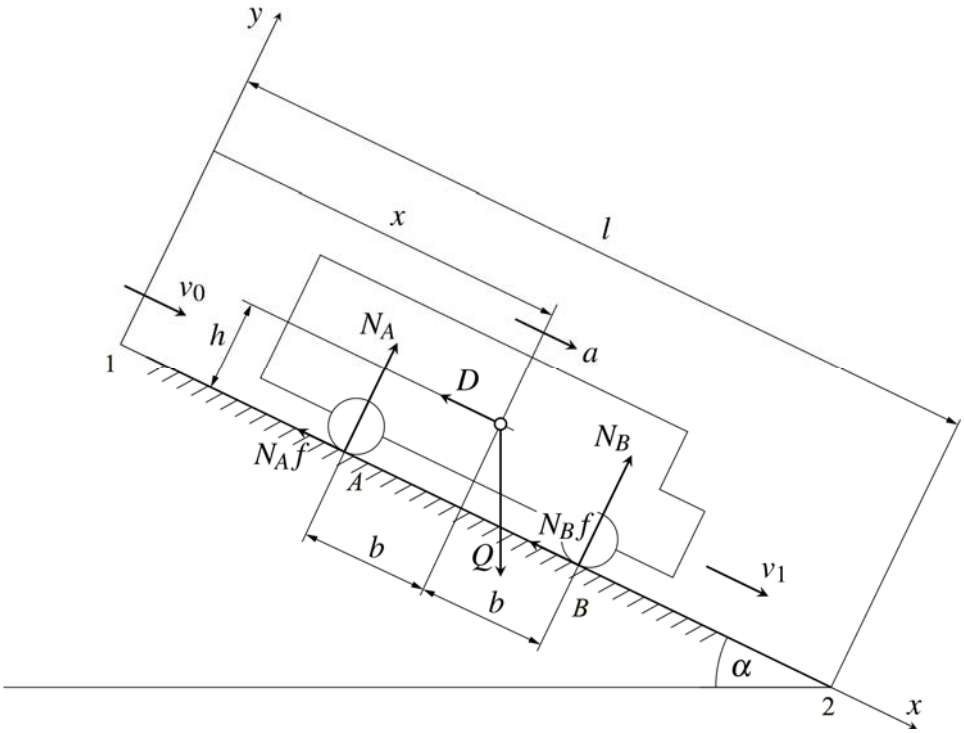


Fig. D40. Motion of a skidding car

Determine: The final velocity v_1 at a distance l from the beginning.

Scalar equations of motion are

$$x: \quad -D - N_A f - N_B f + Q\sin\alpha = 0,$$
$$y: \quad N_A + N_B - Q\cos\alpha = 0,$$
$$M_A: \quad -Qb\cos\alpha - Qh\sin\alpha + 2bN_B + Dh = 0.$$

where $D = ma, \quad Q = mg$.

By subsequent rearranging and integration, we get

$$a = -g(f\cos\alpha - \sin\alpha), \quad \frac{dv^2}{2dx} = g(\sin\alpha - f\cos\alpha), \quad v_1 = \sqrt{v_0^2 + 2gl(\sin\alpha - f\cos\alpha)}.$$

The task has a meaningful solution only if $a > 0$, i.e. if $\sin\alpha > f\cos\alpha$, that is if $\tan\alpha > f$.

Notice that as far as the acceleration is concerned, the result is identical with that of the particle sliding down an inclined plane.

Example – translatory motion.

Given: A rod of weight G is constrained to the frame by two ropes of the same length l. See Fig. D41.

Determine: The equations of motion.

All the points of the rod are subjected to the same trajectory, velocity and acceleration. Thus, by definition, the whole body is subjected to a translatory motion. The equation of motion for the centre of mass, written in a vector form, is $m\mathbf{a} = \mathbf{G} + \mathbf{S}_\mathrm{A} + \mathbf{S}_\mathrm{B}$.

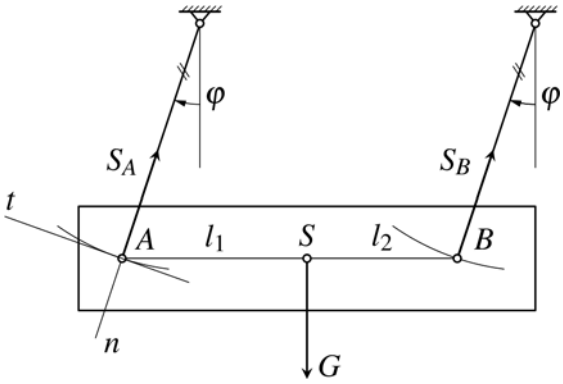


Fig. D41. Swinging rod

Newton's formulation of the equation of motion expressed in a scalar form for tangential and normal components and for the centre of mass S, are

$$t: \quad ml\ddot{\varphi} = -mg\sin\varphi,$$
$$n: \quad ml\dot{\varphi}^2 = S_\mathrm{A} + S_\mathrm{B} - mg\cos\varphi,$$
$$M_\mathrm{S}: \quad 0 = S_A + S_B - mg\cos\varphi.$$

We have used the known kinematic relations for tangent and normal acceleration components i.e, $a_\mathrm{t} = l\ddot{\varphi}, \quad a_\mathrm{n} = l\dot{\varphi}$.

Example – translatory motion

Given: A block of given dimensions and of the weight G slides along the horizontal plane being towed by a constant force P to the right. The coefficient of friction is f. See Fig. D42, where the free body diagram forces are depicted.

Determine: The maximum possible magnitude of the force P which does not cause the block to tilt.

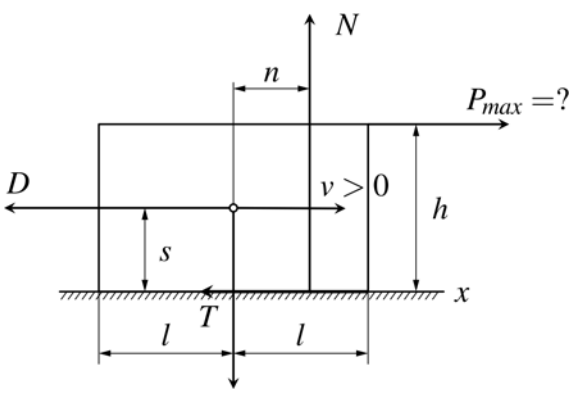


Fig. D42. A Sliding block

The equations of motion, written in d'Alembert' style, are

$$x: \quad P - ma - Nf = 0,$$
$$y: \quad N - G = 0,$$
$$M_\mathrm{S}: \quad Nn - P(h-s) - Nfs = 0.$$

We have three equations for three unknowns, i.e. a, n, N. The 'non-tilt' requirement comes from the fact that the normal reaction should stay within the contact area, thus $n \le l$. From

this we get $P \le G\dfrac{l - fs}{h - s}$. Of course, there is another condition, i.e. h>s, which has to be satisfied.

D8.2 Rotary motion

Summary of kinematics rules for a particle at the radius R subjected to the rotation with angular velocity ω and angular acceleration ε.

The velocity	$v = R\omega$.
The tangential acceleration	$a_t = R\varepsilon = R\dot{\omega}$.
The normal (centripetal) acceleration	$a_n = R\omega^2 = v^2 / R$.

D8.2.1 Planar rotary motion

is described by the fact that the considered body has its symmetry plane perpendicular to the rotation axis. In that case, it is sufficient to write three equations of motion in which inertia effects of individual particles are expressed by three overall effects – by the apparent centrifugal force, by the apparent tangential inertia force and by the apparent inertia moment. D'Alembert style is used for the explanation.

There are three possibilities.

a) Apparent inertia effects (forces and moment) related to the centre of rotation O.
See Fig. D43.

$T = ma_t = mR_S\,\varepsilon$ ... *apparent tangential inertia force*, perpendicular to the OS line, acts at the centre of rotation; its direction is opposite to that of tangent acceleration.

$O = ma_n = mR_S\,\omega^2$ … *apparent normal inertia force* (centrifugal force) acts against the direction of the normal (centripetal) acceleration.

$M_D = I_O\varepsilon$ … *apparent inertia moment* acts against the direction of angular acceleration.

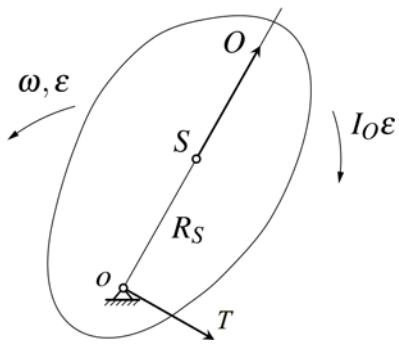


Fig. D43. Apparent inertia forces for the centre of rotation

R_S is the distance between the centre of mass and the axis of rotation and
I_O is the *moment of inertia* about the axis of rotation.

b) Apparent inertia forces related to the centre of mass S. See Fig. D44.

$T = ma_t = mR_S\varepsilon$ ... acting at the centre of mass,

$O = ma_n = mR_S\omega^2$,

$M_D = I_S\varepsilon$.

R_S is the distance between the centre of mass and the axis of rotation, M_D is the apparent inertia moment and I_S is the moment of inertia about the centre of mass.

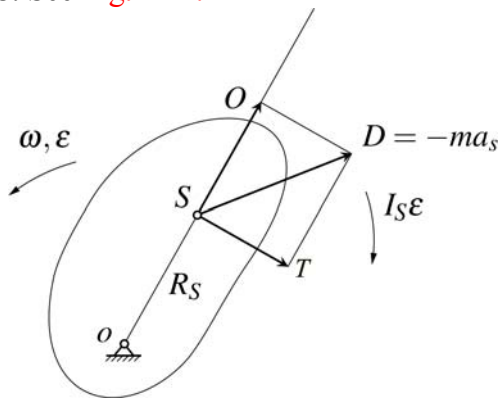


Fig. D44. Apparent inertia forces for the centre of mass

c) The third possibility is rarely used.

It is based on the fact that a force and a moment could be generally replaced by a laterally shifted force. In this case, the centrifugal force is the same as before, but apparent tangential force, whose magnitude is same as before, acts at the distance l from the centre of rotation. Its location is obtained from

$$Tl = I_O\varepsilon \Rightarrow l = I_O\varepsilon / T = I_O\varepsilon / m\rho_S\varepsilon = I_O / m\rho_S .$$

And now, why it is so.

A body of the mass m rotates around the point O by angular velocity ω and by angular acceleration ε. The distance of the centre of mass S from the centre of rotation O is r_S. See Fig. D45.

The apparent normal inertia force, i.e. the centrifugal force, acting on the i-th particle is

$$O_i = m_i r_i \omega^2 .$$

Summing these forces all over the body we get

$$O = \sum O_i = \sum m_i r_i \omega^2 = \omega^2 \underbrace{\int_m r_i \, \mathrm{d}m}_{\text{static moment}} = m r_S \omega^2 .$$

The resulting force is aligned with OS line.

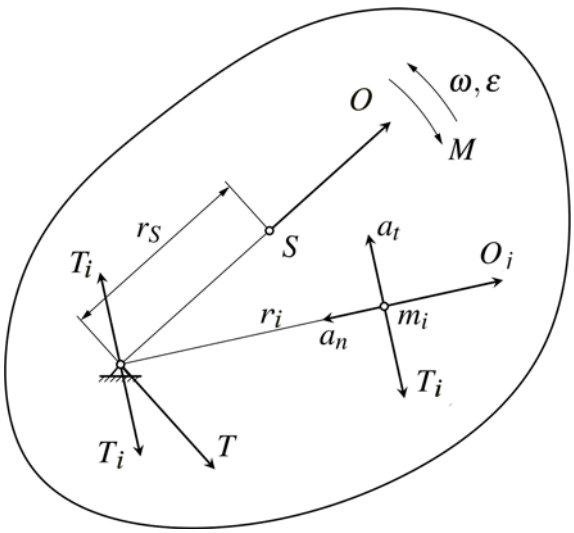


Fig. D45. Resultants of apparent inertia forces

The apparent tangent inertia force acting on the i-th particle is

$$T_i = m_i r_i \varepsilon .$$

When this force is transferred laterally to the origin O, a corresponding couple has to be added, i.e.

$$M_i = T_i r_i = m r_i^2 \varepsilon .$$

Summing it up for the whole body we get

– firstly, the apparent tangential inertia force that acts at the centre of rotation and is perpendicular to OS line

$$T = \sum T_i = \sum m_i r_i \varepsilon = \varepsilon \underbrace{\int_m r_i \, \mathrm{d}m}_{\text{static moment}} = m r_\mathrm{S} \varepsilon ,$$

– and secondly, the apparent inertia moment

$$M = \sum M_i = \sum m_i r_i^2 \varepsilon = \varepsilon \underbrace{\int_m r_i^2 \mathrm{d}m}_{\text{moment of inertia}} = I_\mathrm{O} \varepsilon ,$$

where I_O is the moment of inertia about the centre of rotation.

As before, the resulting tangential force could be laterally shifted to the centre of rotation. Then, the additional couple, i.e. $T r_S$, has to be added. Thus, the apparent inertia moment is

$$M = I_0 \varepsilon - T r_S = \left(I_S + m r_S^2 \right) \varepsilon - m r_S^2 \varepsilon = I_S \varepsilon ,$$

where I_S is the moment of inertia about the centre of mass.

Notice that the directions of apparent inertia effects, in agreement with d'Alembert's principle, always act against the directions of corresponding accelerations.

One has to carefully distinguish two close terminological terms appearing in the relation $M = I\varepsilon$. The term M on the left-hand side is the *apparent inertia moment* – it is measured in $\mathrm{Nm} = \mathrm{kgm^2/s^2}$. On the right-hand side we have the geometrical quantity I which is called the *moment of inertia* – it is measured in $\mathrm{kg/m^2}$. Knowing that the dimension of ε is $1/\mathrm{s}^2$ one is satisfied.

Example – swinging rod

Given: The rod is constrained to the frame by a frictionless joint. In its vertical position, it is held by two initially unstretched springs. The values of string stiffnesses, dimensions, mass and the moment of inertia with respect to the centre of mass, i.e. $c_1, c_2, r_1, r_2, r_S, m, J_S$, are known. In Fig. D46 the rod is depicted in a generic position, characterized by the angle φ, with corresponding free body diagram forces and moments.
Determine: Period of vibration under the assumption of small rotational displacements.

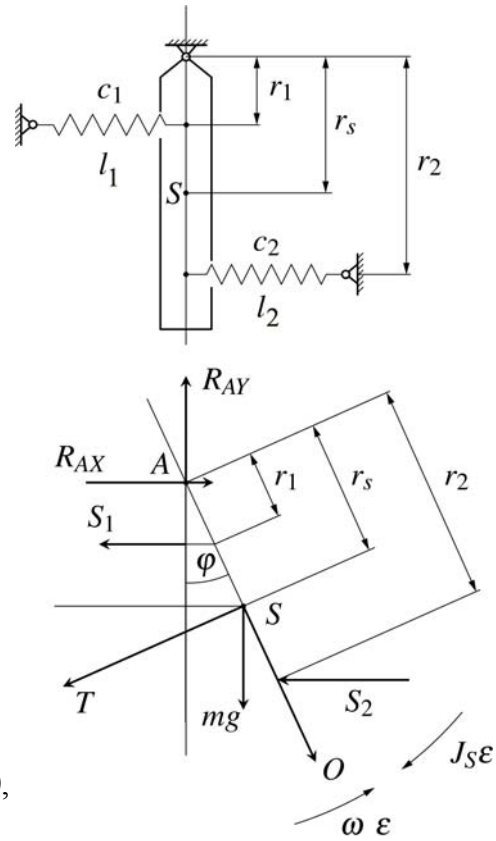


Fig. D46. Rotating rod

If apparent inertia effects are considered with respect to the centre of mass S, then the scalar equations of motion are

$$x: \quad R_{Ax} - S_1 - S_2 - T\cos\varphi + O\sin\varphi = 0,$$
$$y: \quad R_{Ay} - mg - T\sin\varphi - O\cos\varphi = 0,$$
$$M_A: \quad Tr_S + J_S\varepsilon + S_1 r_1 \cos\varphi + S_2 r_2 \cos\varphi + mgr_S \sin\varphi = 0,$$

where centrifugal and apparent tangent inertia forces, spring forces and the relations between the moment of inertia with respect to point A and to the centre of mass, are

$$O = mr_S\omega^2 = mr_S\dot{\varphi}^2, T = mr_S\varepsilon = mr_S\ddot{\varphi},$$
$$S_1 = c_1 r_1 \sin\varphi, S_2 = c_2 r_2 \sin\varphi,$$
$$J_A = mr_S^2 + J_S.$$

The last relation is sometimes referred to as the *parallel axis theorem* or the *Steiner's rule*.

Note: When elongations of spring forces S_1, S_2 are evaluated a small arc due to the rod rotation is approximated by a straight line. For small angles, this is an acceptable approximation.

The third equation leads to

$$\ddot{\varphi}\underbrace{(mr_S + J_S)}_{J_A} + (c_1 r_1^2 + c_2 r_2^2)\sin\varphi\cos\varphi + mgr_S \sin\varphi = 0.$$

For small angles, we use the following approximations, i.e. $\sin\varphi \cong \varphi$, $\cos\varphi \cong 1$, so

$$J_A\ddot{\varphi} + \left(c_1 r_1^2 + c_2 r_2^2\right)\varphi + mgr_S\varphi = 0,$$

$$\ddot{\varphi} + \underbrace{\frac{c_1 r_1^2 + c_2 r_2^2 + mgr_S}{J_A}}_{\Omega^2}\varphi = 0.$$

And finally, the period of vibration is

$$T = \frac{2\pi}{\Omega} = \sqrt{\frac{J_A}{c_1 r_1^2 + c_2 r_2^2 + mgr_S}}.$$

Example – falling rod

Given: r_S, m, g, J_A, S ... the centre of gravity. Fig. D47.
Determine: How the rod bar falls from the vertical position, i.e. find the function $\omega = \omega(\varphi)$.

Equations of motion are

$$x: \quad R_{Ax} - T\cos\varphi + O\sin\varphi = 0,$$
$$y: \quad R_{Ay} + T\sin\varphi + O\cos\varphi - mg = 0,$$
$$M_A: \quad mgr_S\sin\varphi - J_A\varepsilon = 0.$$

Normal and tangential accelerations are

$$a_n = r_S\omega^2, \quad a_t = r_S\varepsilon.$$

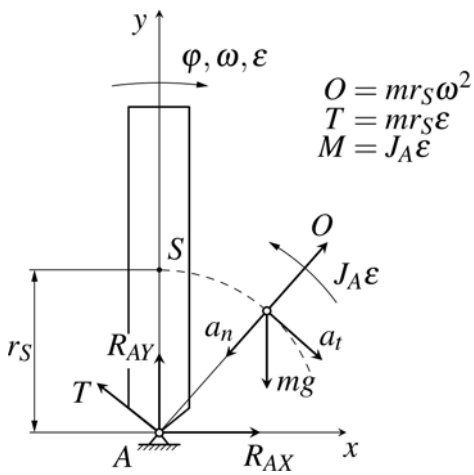


Fig. D47. Falling rod

Apparent normal (centrifugal) and tangential inertia forces and inertia moment are

$$O = ma_n = mr_S\omega^2,$$
$$T = ma_t = mr_S\varepsilon,$$
$$M = J_A\varepsilon. \qquad \text{Recall that } J_A = J_S + mr_S^2.$$

Kinematic relations

$$\omega = \frac{\mathrm{d}\varphi}{\mathrm{d}t}, \quad \varepsilon = \frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\mathrm{d}\omega^2}{2\mathrm{d}\varphi}.$$

From the moment equation of motion, we subsequently get

$$\varepsilon = \frac{mgr_{\mathrm{S}}\sin\varphi}{J_{\mathrm{A}}}, \qquad \frac{\mathrm{d}\omega^2}{2\mathrm{d}\varphi} = \frac{mgr_{\mathrm{S}}\sin\varphi}{J_{\mathrm{A}}},$$

$$\int_0^{\omega^2}\mathrm{d}\omega^2 = \frac{2mgr_{\mathrm{S}}}{J_{\mathrm{A}}}\int_0^{\varphi}\sin\varphi\,\mathrm{d}\varphi, \qquad \omega^2 = -\frac{2mgr_{\mathrm{S}}}{J_{\mathrm{A}}}(\cos\varphi - 1).$$

So the angular velocity ω, expressed as a function of angle φ, is

$$\omega = \sqrt{\frac{2mgr_{\mathrm{S}}}{J_{\mathrm{A}}}(1-\cos\varphi)}.$$

D8.2.2 Spatial rotation of a body about an axis

The coordinate system ξ,η,ζ is firmly connected to the rotating body. See Fig. D48. A generic mass particle m_i, subjected to the rotation around the ξ axis by the angular velocity ω and by the angular acceleration ε, has the normal acceleration $a_{in} = \rho_i\omega^2$ and the tangent acceleration $a_{it} = \rho_i\varepsilon$. According to d'Alembert principle, there are the apparent centrifugal force $O_i = m_i\rho_i\omega^2$ and apparent tangential inertia force $T_i = m_i\rho_i\varepsilon$. The directions of forces O_i, T_i are opposite to the directions of corresponding accelerations $a_{\mathrm{n}_i}, a_{\mathrm{t}_i}$.

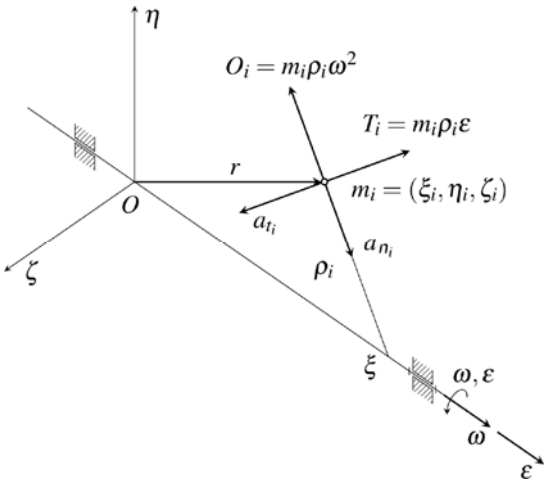


Fig. D48. Spatial rotation

In the chapter devoted to kinematics, we have derived

$$a_{i\xi} = 0,$$
$$a_{i\eta} = -\omega^2\eta_i - \varepsilon\zeta_i,$$
$$a_{i\zeta} = -\omega^2\zeta_i + \varepsilon\eta_i.$$

It should be reminded why it is so. The velocity of a particle defined by the radius vector $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ of a body subjected to rotation defined by the angular velocity $\vec{\omega} = \omega_x\vec{i} + \omega_y\vec{j} + \omega_z\vec{k}$ is given by the cross product

$$\vec{v} = \vec{\omega}\times\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix} = \vec{i}\underbrace{(\omega_y z - \omega_z y)}_{v_x} + \vec{j}\underbrace{(\omega_z x - \omega_x z)}_{v_y} + \vec{k}\underbrace{(\omega_x y - \omega_y x)}_{v_z}.$$

The acceleration components are obtained by expressing the derivatives of velocity components with respect to time

$$a_x = \frac{\mathrm{d}v_x}{\mathrm{d}t} = \dot{\omega}_y z + \omega_y \dot{z} - \dot{\omega}_z y - \omega_z \dot{y} = \varepsilon_y z + \omega_y v_z - \varepsilon_z y - \omega_z v_y ,$$
$$a_y = \varepsilon_z x + \omega_z v_x - \dot{\varepsilon}_x z - \omega_x v_z = \varepsilon_z x + \omega_z \underbrace{\left(\omega_y z - \omega_z y\right)}_{v_x} - \dot{\varepsilon}_x z - \omega_x \underbrace{\left(\omega_x y - \omega_y x\right)}_{v_z},$$
$$a_z = \varepsilon_x y + \omega_x v_y - \dot{\varepsilon}_y x - \omega_y v_x = \varepsilon_x y + \omega_x \underbrace{\left(\omega_z x - \omega_x z\right)}_{v_y} - \dot{\varepsilon}_y x - \omega_y \underbrace{\left(\omega_y x - \omega_z y\right)}_{v_x}.$$

In our case, we have $\omega_y = \omega_z = \varepsilon_y = \varepsilon_z = 0$, so

$$a_x = 0 ,\ a_y = -\omega_x^2 y - \varepsilon_x z ,\ a_z = -\omega_x^2 z + \varepsilon_x y .$$

The similarity is obvious. It suffices to rename variables in such a way that

$$x \to \xi, y \to \eta, z \to \zeta .$$

Now, back to the resulting force which would arise due to summation of elementary forces O_i, T_i. These forces could be expressed by components in ξ, η, ζ directions as

$$D_{i\eta} = -m_i a_{i\eta} ,$$
$$D_{i\zeta} = -m_i a_{i\zeta} , \text{ where}$$
$$a_{i\eta} = -\omega^2 \eta_i - \varepsilon \zeta_i ,$$
$$a_{i\zeta} = -\omega^2 \zeta_i + \varepsilon \eta_i .$$

Using the definition of static moments, coordinates of the centre of mass η_S, ζ_S and the overall mass of the body m we get the apparent inertia forces in the form

$$D_\eta = -\sum m_i a_{i\eta} = \sum m_i \omega^2 \eta_i + \sum m_i \varepsilon \zeta_i = \omega^2 m \eta_S + \varepsilon m \zeta_S , \qquad \text{(D8_7)}$$
$$D_\zeta = -\sum m_i a_{i\zeta} = -\sum m_i \left(-\omega^2 \zeta_i + \varepsilon \eta_i\right) = \omega^2 m \zeta_S - \varepsilon m \eta_S . \qquad \text{(D8_8)}$$

The above relations could be easily verified by a simple geometric consideration depicted in Fig. D49. The projections of elementary forces O_i, T_i into the coordinate axes are

$$O_{i\eta} = O_i \cos\varphi_i = m_i \omega^2 \rho_i \cos\varphi_i = m_i \omega^2 \eta_i ,$$
$$O_{i\zeta} = O_i \sin\varphi_i = m_i \omega^2 \rho_i \sin\varphi_i = m_i \omega^2 \zeta_i ,$$
$$T_{i\eta} = T_i \sin\varphi_i = m_i \varepsilon \rho_i \sin\varphi_i = m_i \varepsilon \zeta_i ,$$
$$T_{i\zeta} = T_i \cos\varphi_i = m_i \varepsilon \rho_i \cos\varphi_i = m_i \varepsilon \eta_i .$$

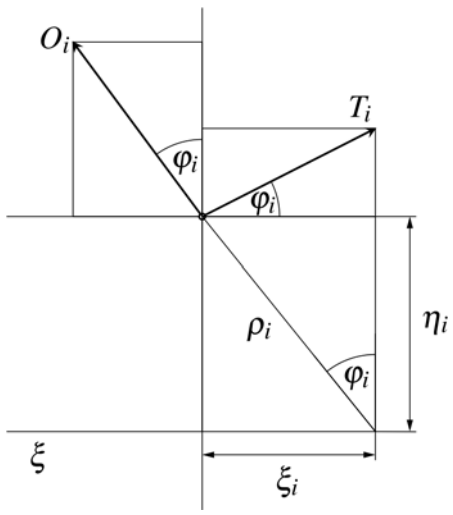


Fig. D49. Components of apparent forces

The forces were summed up and transferred into the origin of the coordinate system. This requires a few moment components to be added.

The moment of apparent inertia forces is

$$-\sum \vec{r}_i \times m_i \vec{a} = -\sum m_i \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \xi_i & \eta_i & \zeta_i \\ a_{i\xi} & a_{i\eta} & a_{i\zeta} \end{vmatrix}.$$

The components of this apparent vector related to the coordinate axes are

$$\begin{aligned} &\xi: \quad -\sum m_i \left(\eta_i a_{i\zeta} - \zeta_i a_{i\eta}\right), \\ &\eta: \quad -\sum m_i \left(\zeta_i a_{i\xi} - \xi_i a_{i\zeta}\right), \\ &\zeta: \quad -\sum m_i \left(\xi_i a_{i\eta} - \eta_i a_{i\xi}\right), \end{aligned}$$

where $\sum m_i \eta_i, \sum m_i \zeta_i, \sum m_i \xi_i$ are static moments about coordinate axes.

Substituting

$$\begin{aligned} &a_{i\xi} = 0, \\ &a_{i\eta} = -\omega^2 \eta_i - \varepsilon \zeta_i, \\ &a_{i\zeta} = -\omega^2 \zeta_i + \varepsilon \eta_i, \end{aligned}$$

into previously derived relations we get three moments.

1) The moment of apparent inertia forces about the ξ axis

$$\begin{aligned} M_\xi &= -\sum m_i \left(\eta_i a_{i\zeta} - \zeta_i a_{i\eta}\right) = -\sum m_i \left[\eta_i \left(-\omega^2 \zeta_i + \varepsilon \eta_i\right) - \zeta_i \left(-\omega^2 \eta_i - \varepsilon \zeta_i\right)\right] = \\ &-\varepsilon \sum m_i \left(\eta_i^2 + \zeta_i^2\right) = -\varepsilon \sum m_i \rho_i^2 = -I_\xi \varepsilon. \end{aligned} \qquad \text{... (D8\_9)}$$

2) The moment of apparent inertia forces about the η axis

$$M_\eta = -\sum m_i \left[\zeta_i 0 - \xi_i \left(-\omega^2 \zeta_i + \varepsilon \eta_i\right)\right] = -\omega^2 \sum m_i \xi_i \zeta_i + \varepsilon \sum m_i \xi_i \eta_i = -\omega^2 C_\eta + \varepsilon C_\zeta \quad \text{... (D8\_10)}$$

3) The moment of apparent inertia forces about the ζ axis

$$M_\zeta = -\sum m_i \left[\xi_i \left(-\omega^2 \eta_i - \varepsilon \zeta_i\right) - 0\right] = \omega^2 \sum m_i \xi_i \eta_i + \varepsilon \sum m_i \xi_i \zeta_i = \omega^2 C_\zeta + \varepsilon C_\eta. \qquad \text{... (D8\_11)}$$

The quantity I_ξ is the moment of inertia about the rotation axis.

If $I_\xi = 0$, we say that the body is statically balanced. It means that the centre of rotation of that body 'sits' at the axis of rotation.

The quantities $C_\xi = C_{\eta\zeta}$ a $C_\eta = C_{\xi\zeta}$ are deviatoric moments.

A dynamically balanced body requires the deviatoric moments to be identically equal to zero as well.

D8.3. General planar motion

We proceed the same way as in kinematics and complement each acceleration component with a corresponding apparent inertia force.

In a given moment the motion of a generic particle of a body subjected to general planar motion is assumed to be described by the velocity and acceleration of the reference point plus by a relative rotational velocity and acceleration of the considered particle around the reference point. In kinematics, we have described the basic and the Coriolis decomposition.

D8.3.1. Basic decomposition

It is advantageous to decompose the overall motion into two parts, i.e. the carrier motion of the translatory nature plus the relative rotational motion around the chosen reference point.

There are two ways how to proceed.

First. The decomposition is carried out with respect a generic reference point K, whose trajectory, velocity $\vec{v}_K$ and acceleration $\vec{a}_K$, as well as relative angular velocity ω and relative angular acceleration, are known. See Fig. D50.

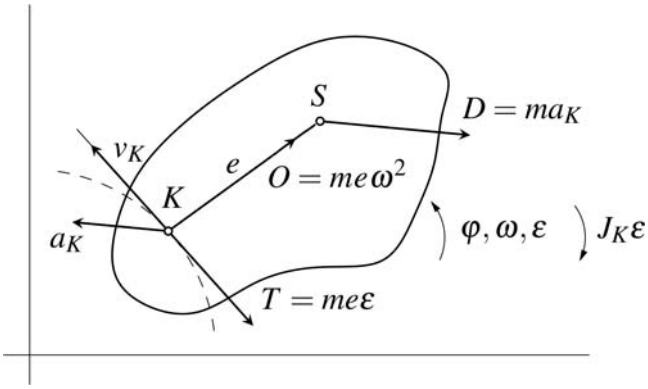


Fig. D50. Dynamics of general planar motion – 1

Then, the magnitudes of apparent inertia forces and of apparent inertia moment are

$$D = ma_K \qquad \text{(D8_12)}$$

… the apparent inertia force due to the carrier translatory motion, it is situated in the center of mass S of the considered body, its direction is opposite to that of the carrier acceleration $\vec{a}_K$,

$$O = me\omega^2 \qquad \text{(D8_13)}$$

… the apparent relative normal force (called the centrifugal force), situated in the reference point, its direction is opposite to that of relative normal acceleration of the centre of mass S. The quantity e is the shortest distance between the centre of rotation and the centre of mass.

$$T = me\varepsilon \qquad \text{(D8\_14)}$$

… the apparent relative tangential force, situated in the reference point, its direction is opposite to that of relative tangential acceleration of the centre of mass S.

$$M = J_K \varepsilon \qquad \text{(D8\_15)}$$

… the apparent relative inertia moment, its direction is opposite to that of relative angular acceleration ε, where J_K is the moment of inertia of the body with respect to the reference point K.

Second. The situation is simplified if the centre of mass is chosen as the reference point. See Fig. D51.

In this case, the forces O and T become null, since the distance e is zero. What remains is the apparent inertia force due to the translatory carrier motion. The magnitude of this force is

$$D = ma_S \qquad \text{(D8\_16)}$$

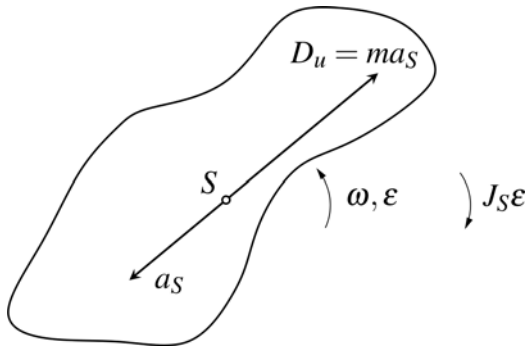


Fig. D51. Dynamics of general planar motion – 2

and the apparent inertia moment due to the relative rotation. The magnitude of this moment is

$$M = J_S \varepsilon, \qquad \text{(D8\_17)}$$

where J_S is the moment of inertia of the considered body with respect to the centre of mass.

Example – cylinder rolling down an inclined plane

Given: mass m, moment of inertia J_S with respect to the center of mass S, radius r, angle α.
Determine: equations of motion

In Fig. D52 is depicted a cylinder at a generic position x, the immediate quantities are the velocity v, acceleration of the centre of cylinder a.

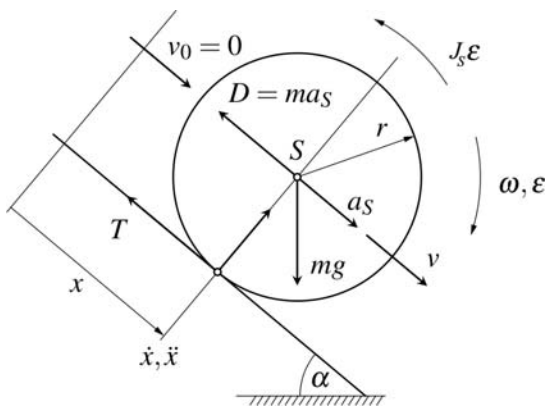


Fig. D52. Rolling cylinder

A free body forces are the normal force N, the weight mg, the rolling resistance force T. It should be reminded that the rolling resistance force differs from the force of friction.

The apparent inertia effects consist of the force D due to the translatory motion and of the apparent inertia moment $J_S\varepsilon$ whose direction is against that of angular acceleration ε.

We can write the kinematic relations in the form
$v_S = r\omega, \quad a_S = r\varepsilon$

and then the equations of motion are

$$
\begin{aligned}
x: &\quad mg\sin\alpha - T - ma_S = 0, \\
y: &\quad -mg\cos\alpha + N = 0, \\
M_S: &\quad Tr - J_S\varepsilon = 0.
\end{aligned}
$$

$$\Rightarrow T, N, \varepsilon, a_S.$$

When the unknowns are calculated, the condition of pure rolling has to be checked. The rolling resistance should be always less than the force of friction, i.e. $T < Nf$, where f is the coefficient of friction.

The rolling of a body might be imagined as a combination of a translatory motion of the body, characterized by the translatory motion of the centre of mass, plus the rotary motion of the body around the centre of mass. The kinetic energy is obtained by summing the translatory and rotary energy contributions, thus

$$E = \frac{1}{2}mv_S^2 + \frac{1}{2}J_S\omega^2.$$

This expression is sometimes referred to as the König's rule.

The velocity at the location x might be alternatively computed from the condition that the difference of kinetic energies (at the end minus that at the beginning) is equal to the work exerted by external forces. It is only the body's weight which works here.

$$
\begin{aligned}
&E_x - E_{x=0} = W, \\
&\frac{1}{2}mv_S^2 + \frac{1}{2}J_S\omega^2 - 0 = mgx\sin\alpha, \\
&\frac{1}{2}(m + \frac{J_S}{r^2})v_S^2 = mgx\sin\alpha \quad \Rightarrow \quad v_S.
\end{aligned}
$$

D8.3.2. Coriolis decomposition

The general planar motion could also be decomposed into the carrier motion of rotary motion plus the relative motion which could be of translatory or rotary nature.

Example – a pendulum on the merry-go-round.

A pendulum is attached at joint A to the rotating frame. See Fig. D53.

Given: ω_{21}, r, l, m
A rod, attached to an arm rotating by a constant angular velocity $\omega_{21} = const$, could freely swing about the joint A.

Determine: Apparent inertia forces
The motion (31) of the particle of mass m at point S can be decomposed into the carrier rotation (21) plus the relative rotation (32).

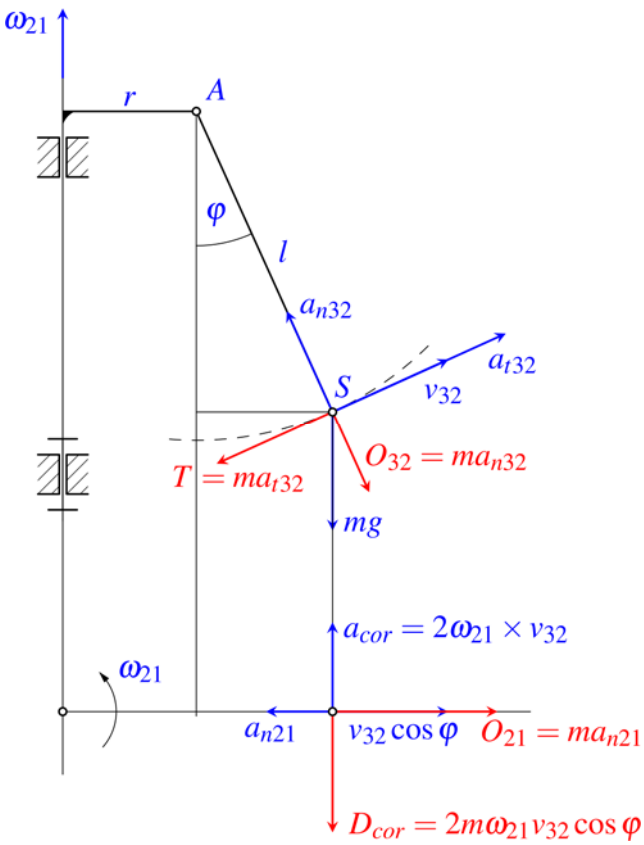


Fig. D53. Pendulum attached to merry go round

Kinematics – velocities and accelerations.

Velocitity of S with respect to the frame (1)

S: $\vec{v}_{31} = \vec{v}_{32} + \vec{v}_{21}$,

where the magnitudes of velocities are

$$v_{32} = l\dot{\varphi}, \qquad v_{12} = (r + l\sin\varphi)\omega_{21}.$$

Acceleration of S with respect to the frame (1)

S: $\vec{a}_{31} = \vec{a}_{32} + \vec{a}_{21} + \vec{a}_{\text{cor}}$,

$$\vec{a}_{32} = \vec{a}_{32\text{t}} + \vec{a}_{32\text{n}}, \qquad \vec{a}_{21} = \vec{a}_{21\text{t}} + \vec{a}_{21\text{n}}, \qquad \vec{a}_{\text{cor}} = 2\vec{\omega}_{21} \times \vec{v}_{32},$$

where the magnitudes of accelerations are

$a_{32\mathrm{t}} = l\ddot{\varphi}$, $a_{32\mathrm{n}} = l\dot{\varphi}^2$, $a_{21\mathrm{t}} = 0$, $a_{21\mathrm{n}} = (r + l\sin\varphi)\omega_{21}^2$, $a_{\mathrm{cor}} = 2\omega_{21}v_{32}\sin\left(\frac{\pi}{2} - \varphi\right) = 2\omega_{21}l\dot{\varphi}\cos\varphi$.

Dynamics – vectors of apparent inertia forces and their magnitudes.

$\vec{O}_{32} = -m\vec{a}_{32\mathrm{n}}$, $O_{32} = ml\dot{\varphi}^2$ ... apparent normal inertia (centrifugal) force due to relative rotation,

$\vec{T}_{32} = -ma_{32\mathrm{t}}$, $T_{32} = ml\ddot{\varphi}$ ... apparent tangential inertia force due to relative rotation,

$\vec{O}_{21} = -ma_{21\mathrm{n}}$,

... $O_{21} = m(r + \sin\varphi)\omega_{21}^2$ ... apparent normal inertia (centrifugal) force due to carrier rotation,

$\vec{T}_{21} = \vec{0}$, $T_{21} = 0$, ($\varepsilon_{21} = 0$) ... apparent tangential inertia force due to carrier rotation.

And finally

$\vec{D}_{\mathrm{cor}} = -m\vec{a}_{\mathrm{cor}}$, $D_{\mathrm{cor}} = 2\omega_{21}l\dot{\varphi}\cos\varphi$ ... apparent Coriolis inertia force.

D8.4. Summary to dynamics of rigid bodies

At first, consider a system of individual particles of mass m_i, later we will deal with elementary mass elements $\mathrm{d}m$.

Let x, y, z is an inertial coordinate system and ξ, η, ζ is another coordinate system which translates and rotates with respect to the former. The origin of ξ, η, ζ system is defined by a radius vector $\mathbf{r}_\Omega$. The system ξ, η, ζ rotates with respect to x, y, z with the angular velocity $\boldsymbol{\omega}$ and the angular acceleration $\boldsymbol{\varepsilon}$.

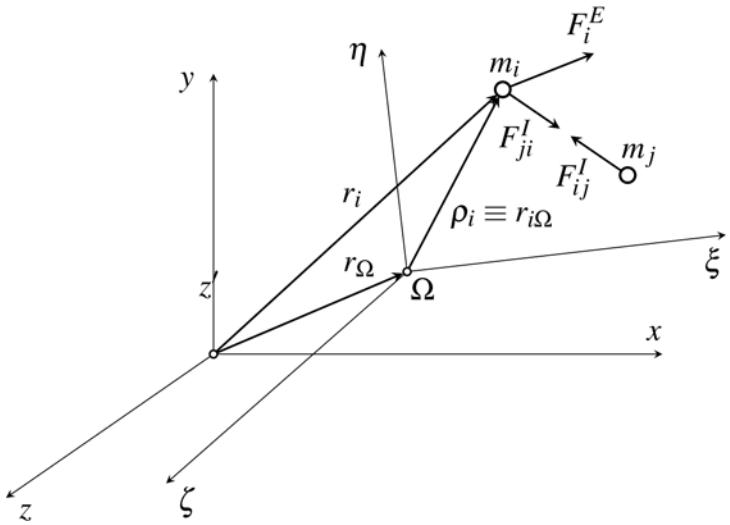


Fig. D54. External and internal forces

External and internal forces acting on the particle m_i are $\mathbf{F}_i^{\mathrm{E}}, \mathbf{F}_i^{\mathrm{I}}$.
Radius vector ρ_i determines the location of m_i with respect to the origin of the coordinate system ξ, η, ζ. See Fig. D54.

Equations of motion are

$$\sum m_i \mathbf{a}_i = \sum \mathbf{F}_i^{\mathrm{E}}, \qquad \text{(D8\_18)}$$

$$\sum \mathbf{r}_i \times m_i \mathbf{a}_i = \sum \mathbf{r}_i \times \mathbf{F}_i^{\mathrm{E}}. \qquad \text{(D8\_19)}$$

In kinematics, we have derived that the velocity and acceleration of the i-th particle can be expressed by

$$\mathbf{v}_i = \mathbf{v}_\Omega + \boldsymbol{\omega} \times \boldsymbol{\rho}_i + \mathbf{v}_{i\text{relative}} = \mathbf{v}_{i\text{carrier}} + \mathbf{v}_{i\text{relative}} \quad \text{... carrier and relative velocity,} \qquad \text{(D8\_20)}$$

$$\mathbf{a}_i = \underbrace{\mathbf{a}_\Omega + \underbrace{\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_i}_{\mathbf{a}_{i\text{tangential}}} + \underbrace{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_i)}_{\mathbf{a}_{i\text{normal}}}}_{\mathbf{a}_{i\text{carrier}}} + \underbrace{2(\boldsymbol{\omega} \times \mathbf{v}_{i\text{relative}})}_{\mathbf{a}_{i\text{Coriolis}}} + \mathbf{a}_{i\text{relative}} = \mathbf{a}_{i\text{carrier}} + \mathbf{a}_{i\text{Coriolis}} + \mathbf{a}_{i\text{relative}} . \qquad \text{(D8\_21)}$$

... carrier, Coriolis and relative accelerations.

Indices relative, carrier denote relative and carrier components, respectively. Substituting Eq. (D8_21) into Eq. (D8_18) we get

$$\sum m_i (\mathbf{a}_{i\text{carrier}} + \mathbf{a}_{i\text{Coriolis}} + \mathbf{a}_{i\text{relative}}) = \sum \mathbf{F}_i^{\text{E}}$$

and after introducing apparent forces the equations of motion have

$$\mathbf{0} = \sum \mathbf{F}_i^{\text{E}} - \sum m_i (\mathbf{a}_{i\text{carrier}} + \mathbf{a}_{i\text{Coriolis}} + \mathbf{a}_{i\text{relative}}) = \sum \mathbf{F}_i^{\text{E}} + \sum \mathbf{D}_{i\text{carrier}} + \sum \mathbf{D}_{i\text{Coriolis}} + \sum \mathbf{D}_{i\text{relative}}$$

... (D8_22)

Evidently, we have introduced

$\mathbf{D}_{i\text{u}} = -m_i \mathbf{a}_{i\text{carrier}}$ … apparent forces due to carrier motion,

$\mathbf{D}_{i\text{r}} = -m_i \mathbf{a}_{i\text{relative}}$ … apparent forces due to relative motion,

$\mathbf{D}_{i\text{C}} = -m_i \mathbf{a}_{i\text{Coriolis}}$ … apparent forces due to Coriolis acceleration.

Moment effects are obtained by substituting (D8_21) into (D8_19).

$$\mathbf{0} = \sum \mathbf{r}_i \times \mathbf{F}_i^{\text{E}} + \sum \mathbf{r}_i \times \mathbf{D}_{i\text{carrier}} + \sum \mathbf{r}_i \times \mathbf{D}_{i\text{relative}} + \sum \mathbf{r}_i \times \mathbf{D}_{i\text{Coriolis}} . \qquad \text{(D8\_23)}$$

Expressed in words

Writing equations of motion in a non-inertial coordinate system requires adding apparent inertia forces due to the carrier, relative and Coriolis accelerations.

If the carrier motion of translatory nature, then there are no Coriolis forces since $\boldsymbol{\omega} = \boldsymbol{\varepsilon} = \mathbf{0}$.

Example – dynamical balancing

In Fig. D55 there is depicted a machine part originally consisting of two cylindrical shafts, denoted by number 1 and 2. The first has the length a and the diameter d_1 while the second has the length b and the diameter d_2. To that part, which is dynamically balanced, two additional small cylinders, denoted by numbers 3 and 4, are attached. The rotating body is supported by two bearings – radial on the left, axi-radial on the right. There are two

coordinate systems. One stationary, the other rotating with the body, the latter is distinguished by primes.

Given: Dimensions, ω, ε.

Determine: The magnitudes and positions of two counterweights to be added in order to secure the dynamical balancing of the depicted machine part. The counterweight masses should be positioned in planes I and II, respectively. See Fig. D55.

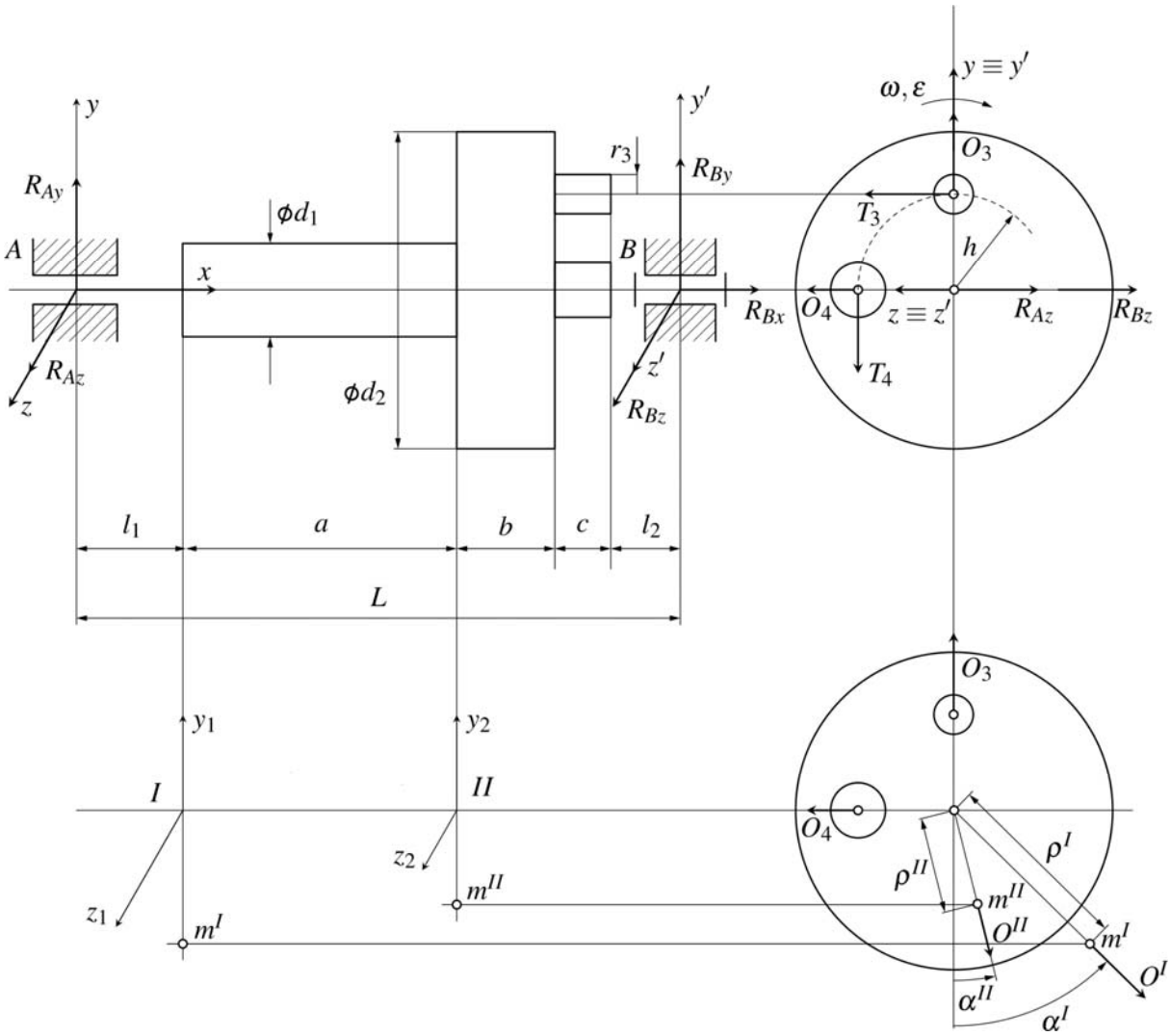


Fig. D55. Dynamical balancing

The cylinders 3 and 4 are considered as particles. Then, the tangential and centrifugal apparent inertia forces, acting on them during the rotation with angular velocity ω and angular acceleration ε, are

$$T_3 = m_3 h\varepsilon, \quad O_3 = m_3 h\omega^2$$
$$T_4 = m_4 h\varepsilon, \quad O_4 = m_4 h\omega^2 .$$

The scalar equations of motion are

$$
\begin{aligned}
&M_x: \quad M = J_x \varepsilon, \\
&x: \quad R_{\mathrm{B}x} = 0, \\
&M_y: \quad R_{\mathrm{B}z} L - O_4 (L - l_2 - c/2) - T_3 (L - l_2 - c/2) = 0, \\
&M_z: \quad O_3 (L - l_2 - c/2) - T_4 (L - l_2 - c/2) + R_{\mathrm{B}y} L = 0, \\
&M_{z'}: \quad R_{\mathrm{A}y} L - T_4 (l_2 + c/2) - O_3 (l_2 + c/2) = 0, \\
&M_{y'}: \quad R_{\mathrm{A}z} L - T_3 (l_2 + c/2) - O_4 (l_2 + c/2) = 0.
\end{aligned}
$$

We intend to balance the body by adding two so-called counterweights in the form of two mass particles in planes I and II. The balancing particles should be located at distances $\rho^{\mathrm{I}}, \rho^{\mathrm{II}}$ from the rotation axis and oriented by angles $\alpha^{\mathrm{I}}, \alpha^{\mathrm{II}}$ from the vertical plane.

The body is in the state of dynamic equilibrium if the moment effects of apparent inertia forces are null. Let's simplify our effort by assuming that $\omega = const$. Then $\varepsilon = 0 \Rightarrow T = 0$. So, the following conditions have to be satisfied

$$
\begin{aligned}
&M_{y1}: \quad O_4 (a + b + c/2) + O^{\mathrm{II}} a \sin \alpha^{\mathrm{II}} = 0, \\
&M_{y2}: \quad O_4 (b + c/2) - O^{\mathrm{I}} a \sin \alpha^{\mathrm{I}} = 0, \\
&M_{z1}: \quad O_3 (a + b + c/2) - O^{\mathrm{II}} a \cos \alpha^{\mathrm{II}} = 0, \\
&M_{z2}: \quad O_3 (b + c/2) + O^{\mathrm{I}} a \cos \alpha^{\mathrm{I}} = 0,
\end{aligned}
$$

where $O^{\mathrm{I}} = m^{\mathrm{I}} \rho^{\mathrm{I}} \omega^2$ and $O^{\mathrm{II}} = m^{\mathrm{II}} \rho^{\mathrm{II}} \omega^2$. After substitution and cancelling by ω^2 there remain six unknowns, i.e. $m^{\mathrm{I}}, m^{\mathrm{II}}, \rho^{\mathrm{I}}, \rho^{\mathrm{II}}, \alpha^{\mathrm{I}}, \alpha^{\mathrm{II}}$, in previous four equations.. Choosing the values of $\rho^{\mathrm{I}}, \rho^{\mathrm{II}}$, then the remaining four, i.e. $m^{\mathrm{I}}, m^{\mathrm{II}}, \alpha^{\mathrm{I}}, \alpha^{\mathrm{II}}$, could be determined.

D9. References

[1] Truesdell, C. and Toupin, R.: The classical and field theories. Encyclopedia of Physics, Springer, Berlin, 1960.

[2] Lanczos, C.: The variational principles of mechanics. Dover Publications, New York, 1970, ISBN 13-978-0-468-65067-8.
https://archive.org/details/VariationalPrinciplesOfMechanicsLanczos

[3] Feynman, R.: The Feynman Lectures on Physics,
http://www.feynmanlectures.caltech.edu

[4] Philosophiae Naturalis Principia Mathematica by Sir Isaac Newton. Latin text.

[5] Sir Isaac Newton. Philosophiae Naturalis Principia Mathematica as translated by Andrew Motte, 1728.

[6] Dare A. Wells: Theory and Problems of Lagrangian Mechanics, Schaum Publishing Company, New York, 1967.

[7] Guicciardini, N.: Reading the Principia, Cambridge, University Press, 1999, ISBN-13 978-0-521-54403-0.

[8] Pratap, R. and Ruina, A.: Statics and dynamics, Oxford University Press, 2001. www.docdatabase.net/more-introduction-to-statics-dynamics-chapters-1-10-1242197.html.

[9] Kane, T.R.: Dynamics. Holt, Rinehart and Winston, Inc., New York, 1968.

[10] Stejskal, V., Dehombreux P., Eiber, A., Gupta, R., Okrouhlik, M.: *Mechanics with Matlab*, Electronic Textbook , ISBN 2-9600226-2-9, Faculté Polytechnique de Mons, Belgium, April 2001, http://www.geniemeca.fpms.ac.be.

[11] Okrouhlik, M. (editor): *Numerical methods in computational mechanics*. Institute of Thermomechanics, Prague 2009, ISBN 978-80-87012-35-2, http://www.it.cas.cz/files/u1784/Num_methods_in_CM.pdf

[12] Okrouhlik, M.: *Implementation of Nonlinear Continuum Mechanics in Finite Element Codes.* Publications of the Institute of Thermomechanics, Educational Series. ISBN 80-85918-06-4, 256 p. Prague, 1995.

[13] Christoph Schiller: *Motion, Mountain – the adventure of physics*. www.motionmountain.net/author.html.

.

Title	Mechanics of Rigid Bodies
Autor	Miloslav Okrouhlík
Design for	Mechanical engineering students
Publisher	Technical university of Liberec
Approved by	Rector’s office TU of Liberec,
	date 21. 12. 2017, no. RE 75/17
Published	December 2017
No. of pages	202
Edition	1st
First published	2017
No. of publication	55-075-17